3 COUNTABILITY AND CONNECTEDNESS AXIOMS

Definition 3.1 *Let X be a topological space. A subset D of X is* dense *in X iff $\bar{D} = X$.*

X is separable *iff it contains a countable dense subset.*

X satisfies the first axiom of countability or is first countable *iff for each $x \in X$, there is a countable neighbourhood basis at x. Any metrisable space is first countable.*

X satisfies the second axiom of countability or is second countable *iff the topology of X has a countable basis.*

For example, $\mathbb{R}$ is separable, $\mathbb{Q}$ forming a countable dense subset. $\mathbb{R}$ is even second countable, for example the countable collection of open intervals with rational end-points forms a basis.

Lemma 3.2 *Let D be a subset of a space X and let $\mathcal{B}$ be a basis of open sets not containing $\varnothing$. Then D is dense in X iff for each $B \in \mathcal{B}$, $B \cap D \neq \varnothing$.*

Proof. $\Rightarrow$: Suppose that $B \in \mathcal{B}$ but $B \cap D = \varnothing$. Then D is contained in the closed set $X - B$, so $\bar{D} \neq X$.

$\Leftarrow$: Let $x \in X$ and let N be a neighbourhood of x in X. Then there is $B \in \mathcal{B}$ so that $x \in B \subset N$. Since $B \cap D \neq \varnothing$, it follows that $N \cap D \neq \varnothing$, so by Proposition 1.17 $x \in \bar{D}$, so that $\bar{D} = X$. ■

Theorem 3.3 *Let X be second countable. Then X is separable and first countable.*

Proof. Let $\mathcal{B}$ be a countable basis: we may assume that $\varnothing \notin \mathcal{B}$.

For each $B \in \mathcal{B}$, choose $x_B \in \mathcal{B}$. Then by Lemma 3.2, $\{x_B \ / \ B \in \mathcal{B}\}$ is a countable dense subset of X, so X is separable.

For each $x \in X$, $\{B \in \mathcal{B} \ / \ x \in B\}$ is a countable neighbourhood basis at x, so X is first countable. ■

In general no other relations hold between these three properties. For example, an uncountable discrete space is first countable but not separable and an uncountable space with the cofinite topology is separable but not first countable. The real line with the right half-open interval topology is separable and first countable but not second countable.

Theorem 3.4 *The topological product of a countable family of separable (first countable, second countable) spaces is separable (first countable, second countable).*

Proof. Let $\{X_n \ / \ n = 1, 2, \ldots\}$ be a countable family of topological spaces and let $X = \Pi X_n$.

(i) Suppose that for each n, X_n is separable, say $\{x_{i,n} \ / \ i = 1, 2, \ldots\}$ is a countable dense subset. Let $D_m = \{\langle x_{i,n}\rangle \in X \ / \ i = 1 \text{ whenever } n > m\}$. Then D_m is countable, and so also is $D = \cup D_m$. If $U = \Pi U_n$ is a basic open set with $U \neq \varnothing$, then there is m so that for each $n > m$, $U_n = X_n$. For each $n \leq m$, there is $x_{i,n} \in U_n$. Thus $U \cap D_m \neq \varnothing$, so $U \cap D \neq \varnothing$ and hence by Lemma 3.2, D is dense and so X is separable.

(ii) Suppose that for each n, X_n is first countable. Let $x \in X$, and for each n, let $\mathcal{M}_n$ be a countable neighbourhood basis at x_n. We may assume that each member of $\mathcal{M}_n$ is open in X_n. Consider

$$\mathcal{M} = \{\Pi A_n \ / \ A_n = X_n \text{ for all but finitely many } n;\ \text{if } A_n \neq X_n \text{ then } A_n \in \mathcal{M}_n\}.$$

Then $\mathcal{M}$ is a countable family of open neighbourhoods of x. Further, if N is any neighbourhood of x, then $x \in \Pi U_n \subset N$ for some U_n open in X_n with $U_n \neq X_n$ for only finitely many n. If $U_n = X_n$, let $A_n = X_n$, and if $U_n \neq X_n$, then there is $A_n \in \mathcal{M}_n$ so that $x \in A_n \subset U_n$. Then $\Pi A_n \in \mathcal{M}$ and $x \in \Pi A_n \subset \Pi U_n \subset N$. Thus $\mathcal{M}$ is a countable neighbourhood basis at x, so X is first countable.

(iii) Suppose that for each n, X_n is second countable, say $\mathcal{B}_n$ is a countable basis. Then

$$\{\Pi B_n \ / \ B_n = X_n \text{ for all but finitely many } n; \text{ if } B_n \neq X_n \text{ then } B_n \in \mathcal{B}_n\}$$

is a countable basis for X, so X is second countable. ■

As might be expected, we cannot replace the countable product in Theorem 3.4 by an arbitrary product. For example, let A be an uncountable index set and for each $\alpha \in A$, let X_α be the set of positive integers with the discrete topology. Then $\{\{x\} \ / \ x \in X_\alpha\}$ is a countable basis for X_α, so X_α is second countable, and hence separable and first countable.

The space $X = \Pi X_\alpha$ is not first countable, for if $\{N_i\}$ is a countable family of neighbourhoods of $x \in X$, then for each i, $\pi_\alpha(N_i) = X_\alpha$ for all but finitely many indices α. Thus there is $\beta \in A$ so that for each i, $\pi_\beta(N_i) = X_\beta$. Then $\pi_\beta^{-1}(x_\beta)$ is a neighbourhood of x containing no N_i, so $\{N_i\}$ cannot be a neighbourhood basis at x.

Suppose that D is a dense subset of X. Let $\mathcal{P}(D)$ denote the power set of D and define $\varphi : A \to \mathcal{P}(D)$ by $\varphi(\alpha) = D \cap \pi_\alpha^{-1}(1)$. If $\alpha, \beta \in A$ with $\alpha \neq \beta$, then

$$\varphi(\alpha) - \varphi(\beta) = D \cap [\pi_\alpha^{-1}(1) \cap \pi_\beta^{-1}(X_\beta - \{1\})] \neq \varnothing,$$

so φ is injective. Thus card $A \leq$ card $\mathcal{P}(D)$. Hence if card $A > 2^{\aleph_0}$, then card $D > \aleph_0$, i.e. D is uncountable: thus in this case X is not separable.

Definition 3.5 *Let X be a space. Say that a sequence $\langle x_n \rangle$ in X* converges *to $x \in X$ and write $\lim_{n\to\infty} x_n = x$ iff for each neighbourhood N of x, there is n_0 so that for each $n \geq n_0$, $x_n \in N$. If there is n_0 so that for each $n \geq n_0$, $x_n \in N$, say that $\langle x_n \rangle$ is* eventually *in N.*

Convergent sequences are not of as much use in topological spaces as they are in metric spaces. There are two particular problems. Firstly, in a general topological space the limit need not be unique: in fact in an indiscrete space, every sequence converges to every point. Secondly, convergent sequences do not capture the topology near a point: this disadvantage is highlighted in Example 3.9. Nets and filters have been evolved to overcome the main problems but we do not consider either.

Proposition 3.6 *Let X be a Hausdorff space, and let $\langle x_n \rangle$ be a sequence in X. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ then $x = y$.*

Proof. If $x \neq y$ then there are neighbourhoods U of x and V of y so that $U \cap V = \varnothing$. It is impossible for any sequence to be eventually in both U and V. ■

Theorem 3.7 *Let X be a first countable space and let $A \subset X$. Then*

$$\bar{A} = \{x \in X \ / \ \textit{there is a sequence } \langle x_n \rangle \textit{ in } A \textit{ with } \lim_{n\to\infty} x_n = x\}.$$

Proof. If $\langle x_n \rangle$ is a sequence in A and $\lim_{n\to\infty} x_n = x$, then $x \in \bar{A}$ by Proposition 1.17 and the definition of $\lim_{n\to\infty} x_n = x$.

Conversely, suppose that $x \in \bar{A}$. Let $\{N_i \ / \ i = 1, 2, \ldots\}$ be a countable neighbourhood basis at x. We may assume that for each i, $N_{i+1} \subset N_i$; otherwise replace each N_i by $\cap_{j=1}^{i} N_j$. Now for each i, $A \cap N_i \neq \varnothing$; choose $x_i \in A \cap N_i$. Then $\langle x_n \rangle$ is a sequence in A and $\lim_{n\to\infty} x_n = x$. ■

Corollary 3.8 *Let X be a first countable space and let $f : X \to Y$ be a function. Then f is continuous at $x \in X$ iff for every sequence $\langle x_n \rangle$ of points of X such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.*

Proof. $\Rightarrow$: obvious.

$\Leftarrow$: we use the criterion of problem 10 of section 1. Let $A \subset X$ be any set with $x \in \bar{A}$. By Theorem 3.7, there is a sequence $\langle x_n \rangle$ in A for which $\lim_{n\to\infty} x_n = x$. By hypothesis, $\lim_{n\to\infty} f(x_n) = f(x)$. Since $f(x_n) \in f(A)$, we have $f(x) \in \overline{f(A)}$ as required, so f is continuous. ∎

Example 3.9 *The criteria in Theorem 3.7 and Corollary 3.8 are invalid if we remove the hypothesis of first countability.*

For example, let X be any uncountable set and topologise X by the cocountable topology of problem 3 of section 1. Any convergent sequence in X is eventually constant but if A is any proper uncountable subset of X then $\bar{A} \neq A$ so Theorem 3.7 is invalid if we delete first countability. Let Y have the same underlying set as X but with the discrete topology and let $f : X \to Y$ be given by $f(x) = x$. Then f is not continuous although if $\langle x_n \rangle$ is a sequence of points of X and $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} f(x_n) = f(x)$.

Theorem 3.10 *Every regular second countable space is normal.*

Proof. Let X be a regular second countable space, say $\mathcal{B}$ is a countable basis, and suppose that A and B are disjoint closed subsets of X.

For each $x \in A$, the set $X - B$ is a neighbourhood of x, so by regularity of X, there is $U_x \in \mathcal{B}$ so that $x \in U_x \subset \overline{U_x} \subset X - B$. Now $\{U_x \ / \ x \in A\}$ is countable, so we may reindex and rename to $\{U_i \ / \ i = 1, 2, \ldots\}$. Then the countable collection $\{U_i \ / \ i = 1, 2, \ldots\}$ of open sets satisfies: $A \subset \cup_i U_i$; and for each i, $B \cap \overline{U_i} = \varnothing$.

Similarly we can find a countable collection $\{V_i \ / \ i = 1, 2, \ldots\}$ of open sets satisfying: $B \subset \cup_i V_i$; and for each i, $A \cap \overline{V_i} = \varnothing$.

Define the sets Y_i and Z_i $(i = 1, 2, \ldots)$ by $Y_i = U_i - [\cup_{n=1}^{i} \overline{V_n}]$ and $Z_i = V_i - [\cup_{n=1}^{i} \overline{U_n}]$. Then Y_i and Z_i are also open sets. Let $U = \cup_i Y_i$ and $V = \cup_i Z_i$, both open sets. It is claimed that U and V satisfy the requirements of the definition of normality.

$U \cap V = \varnothing$, for if $x \in U \cap V$, then there are i, j so that $x \in Y_i \cap Z_j$. We might as well suppose that $i \geq j$, in which case $x \in Y_i = U_i - [\cup_{n=1}^{i} \overline{V_n}] \subset U_i - \overline{V_j}$ and $x \in Z_j \subset V_j$, a contradiction. Thus $U \cap V = \varnothing$.

$A \subset U$, since $A \subset \cup_i U_i$ and $A \cap \overline{V_j} = \varnothing$. Similarly $B \subset V$.

Thus U and V are disjoint open neighbourhoods of A and B, so X is normal. ∎

Theorem 3.11 (Urysohn's metrisation theorem) *Every second countable T_3 space is metrisable.*

Proof. Let X be a second countable T_3 space, and let $\mathcal{B}$ be a countable basis for X. We construct a metric on X in two steps.

(i) There is a countable family $\{g_n : X \to [0,1] \ / \ n = 1, 2, \ldots\}$ of continuous functions satisfying: for each $x \in X$ and each neighbourhood N of x, there is n so that $g_n(x) > 0$ and $g_n(X - N) = 0$. Indeed by Theorems 3.10 and 2.20, for each pair $U, V \in \mathcal{B}$ for which $\bar{U} \subset V$, there is a continuous function $g_{U,V} : X \to [0,1]$ so that $g_{U,V}(\bar{U}) = 1$ and $g_{U,V}(X - V) = 0$. The collection $\{g_{U,V} \ / \ U, V \in \mathcal{B} \text{ and } \bar{U} \subset V\}$ is countable. Furthermore, if $x \in X$ and N is a neighbourhood of x then there are $U, V \in \mathcal{B}$ so that $x \in U \subset \bar{U} \subset V \subset N$: for such U and V, we have $g_{U,V}(x) > 0$ and $g_{U,V}(X - N) = 0$. Now relabel the countable family $\{g_{U,V} \ / \ U, V \in \mathcal{B} \text{ and } \bar{U} \subset V\}$ as $\{g_n : X \to [0,1] \ / \ n = 1, 2, \ldots\}$.

(ii) Given the family $\{g_n : X \to [0,1] \ / \ n = 1, 2, \ldots\}$ of step (i), define $f_n : X \to [0,1]$ by $f_n(x) = \frac{g_n(x)}{n}$. Now define $d : X \times X \to [0,1]$ by

$$d(x,y) = \text{lub}\{|f_n(x) - f_n(y)| \ / \ n = 1, 2, \ldots\}.$$

It remains to verify that (A) d is a metric; and (B) the topology induced by d is the topology on X.

(A) d is a metric.

(a) $d(x,x) = 0$

(b) If $d(x,y) = 0$ then $x = y$, for if $x \neq y$, then $X - \{y\}$ is a neighbourhood of x so there is n for which $f_n(x) > 0$ and $f_n(y) = 0$. Thus $d(x,y) \geq |f_n(x) - f_n(y)| > 0$.

(c) $d(x,y) = d(y,x)$

(d) $d(x,z) \leq d(x,y) + d(y,z)$, because for each n,

$$|f_n(x) - f_n(z)| \leq |f_n(x) - f_n(y)| + |f_n(y) - f_n(z)| \leq d(x,y) + d(y,z),$$

so $d(x,y) + d(y,z)$ is an upper bound of the set for which $d(x,z)$ is the least upper bound.

(B) the topology induced by d is the topology on X.

(a) let $x \in X$ and let N be a neighbourhood of x. Choose n for which $f_n(x) > 0$ and $f_n(X - N) = 0$, and let $\varepsilon = f_n(x)$. Then for any $y \in X$, if $d(x,y) < \varepsilon$ then $|f_n(x) - f_n(y)| < \varepsilon$ so $f_n(y) > 0$ and hence $y \in N$.
Thus N is also a neighbourhood of x in (X, d).

(b) let $x \in X$ and let $\varepsilon > 0$. Let $m = \left[\frac{1}{\varepsilon}\right]$.
For each $n \leq m$, let $N_n = f_n^{-1}((f_n(x) - \varepsilon, f_n(x) + \varepsilon))$. Then for each $y \in N_n$, $|f_n(x) - f_n(y)| < \varepsilon$. Also, for each $n > m$ we have $\frac{1}{n} < \varepsilon$, so that for each $y \in X$ and each $n > m$, $|f_n(x) - f_n(y)| \leq \frac{1}{n} < \varepsilon$.
Set $N = \cap_{n=1}^{m} N_n$. Then N is a neighbourhood of x since each N_n is; and if $y \in N$, then

$$d(x,y) \leq \max\left\{\frac{1}{m+1}, |f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|, \ldots, |f_m(x) - f_m(y)|\right\} < \varepsilon,$$

so N is contained in the d-ball of radius ε centred at x.

By (a) and (b), the neighbourhood systems for the two topologies coincide. Thus by Proposition 1.11, the two topologies themselves are the same. ∎

Let **2** denote the discrete space whose underlying set is $\{0,1\}$. This space is the prototypical disconnected space.

Definition 3.12 *A space X is* connected *iff every continuous function $f : X \to \mathbf{2}$ is constant. Otherwise the space is* disconnected*, and if X is disconnected then we will call a continuous surjection $\delta : X \to \mathbf{2}$ a* disconnection *(of X).*

A subset C of a space X is connected *iff the subspace C of X is connected.*

Say that two points $x, y \in X$ are connected in *X iff there is a connected subset C of X containing both x and y. The relation "are connected in" is an equivalence relation. Call the equivalence class of $x \in X$ under this relation the* component *of x in X.*

Many proofs involving connectedness make use of a disconnection and argument by reductio ad absurdum. Connectedness and disconnectedness are both obviously topological properties.

Theorem 3.13 *Let X be a space. The following are equivalent:*

(i) X is connected;

(ii) every two points of X are connected in X;

(iii) X cannot be expressed as the union of two non-empty disjoint open subsets;

(iv) X cannot be expressed as the union of two non-empty disjoint closed subsets;

(v) the only subsets of X which are both open and closed are $\varnothing$ and X.

Proof. (i)⇒(ii): trivial.

(ii)⇒(iii): suppose that $X = U \cup V$, where $U \cap V = \varnothing$ and U and V are non-empty open sets. Pick $x \in U$ and $y \in V$. Let $A \subset X$ be any set containing x and y. Then $\delta : A \to \mathbf{2}$ defined by $\delta(A \cap U) = 0$ and $\delta(A \cap V) = 1$ is a disconnection of A, so x and y are not connected in X.

(iii)⇒(iv): if $X = C \cup D$, where $C \cap D = \varnothing$ and C and D are non-empty closed sets, then $X = (X - C) \cup (X - D)$, a union of two non-empty disjoint open sets.

(iv)⇒(v): if S is both open and closed in X but $\varnothing \neq S \neq X$, then S and $X - S$ are two non-empty disjoint closed sets whose union is X.

(v)⇒(i): if $\delta : X \to \mathbf{2}$ is a disconnection of X, then $\delta^{-1}(0)$ is both open and closed but is neither $\varnothing$ nor X. ∎

Theorem 3.14 *Let $\{C_\alpha \ / \ \alpha \in A\}$ be a family of connected subsets of a space X satisfying: there is $\beta \in A$ so that for each α either $C_\alpha \cap \overline{C_\beta} \neq \varnothing$ or $\overline{C_\alpha} \cap C_\beta \neq \varnothing$. Then $C = \cup_{\alpha \in A} C_\alpha$ is connected.*

Proof. Consider firstly the case where $C = X$. Suppose instead that X is not connected. Then by Theorem 3.13, we may write $X = U \cup V$, where U and V are non-empty disjoint open subsets of X. Since C_β is connected, it must lie inside one of U and V, say $C_\beta \subset U$. Then for each α, we must have $C_\alpha \subset U$ for by connectedness, either $C_\alpha \subset U$ or $C_\alpha \subset V$, but if $\alpha \in A$ is such that $C_\alpha \subset V$, then $\overline{C_\beta} \subset \bar{U} \subset X - V$, so $C_\alpha \cap \overline{C_\beta} = \varnothing$ and $\overline{C_\alpha} \subset \bar{V} \subset X - U$, so $\overline{C_\alpha} \cap C_\beta = \varnothing$, contrary to hypothesis. Thus for each $\alpha \in A$, $C_\alpha \subset U$, so $X \subset U$, and V must be empty, a contradiction. Thus X is connected.

If $C \neq X$ then any point of $C_\alpha \cap \overline{C_\beta}$ must be in C, from which it follows that either $C_\alpha \cap \overline{C_\beta} \neq \varnothing$ or $\overline{C_\alpha} \cap C_\beta \neq \varnothing$ (where closure here refers to closure in C), and hence we may apply the previous case to the subspace C. ∎

Note that a family $\{C_\alpha \ / \ \alpha \in A\}$ of connected subsets having non-empty intersection satisfies the hypotheses of the theorem.

Corollary 3.15 *Suppose that C is a connected subset of X and $A \subset X$ satisfies $C \subset A \subset \bar{C}$. Then A is connected.*

Proof. Apply Theorem 3.14 to the family $\{C\} \cup \{\{x\} \ / \ x \in A\}$. ∎

Theorem 3.16 *For each x in a topological space X, the component of x in X is the largest connected subset of X containing x. Moreover, the component is closed.*

Proof. Let C denote the component of x in X. Clearly if K is any connected subset of X containing x then $K \subset C$. On the other hand, for each $y \in C$, there is a connected set $K_y \subset X$ containing both x and y. By Theorem 3.14, $\cup_{y \in C} K_y$ is connected. Thus $C = \cup_{y \in C} K_y$ is connected, and hence is the largest connected set containing x.

By Corollary 3.15, $\bar{C}$ is connected, so $\bar{C} = C$, i.e. C is closed. ■

Components need not be open. For example, giving the rationals, $\mathbb{Q}$, the usual topology inherited from $\mathbb{R}$, the only connected sets are $\varnothing$ and the singletons, so the only components are the singletons, none of which is open.

Theorem 3.17 *The topological product of connected spaces is connected.*

Proof. The proof proceeds in three stages, considering the product of two spaces, then of finitely many spaces, and finally a product of arbitrarily many spaces.

Let X and Y be connected spaces and let (x_1, y_1) and (x_2, y_2) be any two points of $X \times Y$. Since $\{x_1\} \times Y$ is homeomorphic to Y, it is connected. $X \times \{y_2\}$, being homeomorphic to X, is connected. Moreover, $(x_1, y_2) \in (\{x_1\} \times Y) \cap (X \times \{y_2\})$. Thus by Theorem 3.14, $(\{x_1\} \times Y) \cap (X \times \{y_2\})$ is connected: this set contains both (x_1, y_1) and (x_2, y_2), so by Theorem 3.13(ii), $X \times Y$ is connected.

By induction, a finite product of connected spaces is connected.

Suppose now that $\{X_\alpha \ / \ \alpha \in A\}$ is any family of connected spaces. Let $X = \Pi X_\alpha$, let $x \in X$, and let C be the component of x in X. It is sufficient to show that if U is any non-empty basic open set then $U \cap C \neq \varnothing$, for then by Lemma 3.2, $\bar{C} = X$. However, Theorem 3.16 tells us that C is closed. Thus $C = X$, i.e., X is connected.

Suppose, then, that $U = \Pi U_\alpha \neq \varnothing$, where each U_α is open in X_α and $U_\alpha \neq X_\alpha$ for only finitely many indices α, say $\alpha_1, \ldots, \alpha_n$. For each $i = 1, \ldots, n$, pick $u_i \in U_{\alpha_i}$. Let $y \in X$ be the point defined by $y_\alpha = x_\alpha$ for $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$, and $y_{\alpha_i} = u_i$. Then $y \in U$. Now $K = \{z \in X \ / \ z_\alpha = x_\alpha$ for each $\alpha \in A - \{\alpha_1, \ldots, \alpha_n\}\}$ is homeomorphic to $X_{\alpha_1} \times \ldots \times X_{\alpha_n}$ so is connected by the finite case of the theorem already proven. Further, $x \in K$, so $K \subset C$. Since $y \in K$, we have $y \in U \cap C$, so $U \cap C \neq \varnothing$. ■

Proposition 3.18 *Let $f : X \to Y$ be continuous and let C be a connected subset of X. Then $f(C)$ is connected.*

Proof. If not, then there is a disconnection $\delta : f(C) \to \mathbf{2}$. Then $\delta f : C \to \mathbf{2}$ is also a disconnection. ■

Theorem 3.19 *A subset of $\mathbb{R}$ (usual topology) is connected iff it is an interval.*

Proof. $\Rightarrow$: Let $A \subset \mathbb{R}$. If A is not an interval, then there is $c \in \mathbb{R} - A$ and $a, b \in A$ so that $a < c < b$. Define $\delta : A \to \mathbf{2}$ by $\delta(x) = 0$ for all $x < c$ and $\delta(x) = 1$ for all $x > c$. Then δ is a disconnection of A.

$\Leftarrow$: Let $A \subset \mathbb{R}$ be an interval. If $\delta : A \to \mathbf{2}$ is a disconnection of A, then let $a \in \delta^{-1}(0)$ and $b \in \delta^{-1}(1)$. We may suppose that $a < b$. The non-empty set $B = \{x \in A \ / \ x < b$ and $\delta(x) = 0\}$ is bounded above by b; let $\alpha = \text{lub} B$. By Proposition 1.17, $\alpha \in \overline{B}^A$, where $\overline{B}^A$ denotes the closure of B in A. What is $\delta(\alpha)$?

By Theorem 1.19(vi), $\delta(\overline{B}^A) \subset \overline{\{0\}} = \{0\}$, so $\delta(\alpha) = 0$. Since $\delta(\alpha) = 0$ but $\delta(b) = 1$ we have $(\alpha, b] \neq \varnothing$, so $\delta((\alpha, b]) = \{1\}$, and by Theorem 1.19(vi), $\delta([\alpha, b]) = \delta(\overline{(\alpha, b]}^A) = \overline{\{1\}} = \{1\}$, so $\delta(\alpha) = 1$, a contradiction. ■

Theorem 3.20 (Intermediate Value Theorem) *Let $f : X \to \mathbb{R}$ be continuous, where X is connected. Suppose $x_1, x_2 \in X$ and $y \in \mathbb{R}$ are such that $f(x_1) \leq y \leq f(x_2)$. Then there is $x \in X$ such that $f(x) = y$.*

Proof. By Theorem 3.19, $f(X)$ is an interval, so $y \in f(X)$. ∎

Corollary 3.21 *Let $a, b \in \mathbb{R}$ with $a < b$. Give $[a, b]$ the usual topology, and let $f : [a, b] \to [a, b]$ be continuous. Then there is a point $x \in [a, b]$ such that $f(x) = x$.*

Proof. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = x - f(x)$. Then g is continuous, $g(a) = a - f(a) \leq 0$, and $g(b) = b - f(b) \geq 0$. Thus by Theorem 3.20, there is $x \in [a, b]$ such that $g(x) = 0$, i.e., $f(x) = x$. ∎

Corollary 3.21 is known as *Brouwer's fixed-point theorem* in dimension 1. A point $x \in X$ is a *fixed-point* of a function $f : X \to X$ iff $f(x) = x$. A space X has the *fixed-point property* iff every continuous function $f : X \to X$ has a fixed-point. The fixed-point property is a topological property. Brouwer's fixed-point theorem in general says that $B^n = \{x \in \mathbb{R}^n \ / \ |x| \leq 1\}$ has the fixed-point property; for a proof of this for $n = 2$ see chapter 5.

Using algebraic topology, one can assign to continuous functions on a certain class of topological spaces a number known as the Lefschetz number. The Lefschetz fixed-point theorem says that if this number is non-zero then the function has a fixed-point. It turns out that when the domain and range of the function are B^n then the Lefschetz number is always non-zero.

The solution of ordinary differential equations involves finding a fixed-point of a certain function. For example, if we wish to solve $y' = F(x, y)$ subject to $y = y_0$ when $x = x_0$, then we seek to solve the equation $y = y_0 + \int_{x_0}^{x} F(x, y)dx$. Let X denote the space of all continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ and define $L : X \to X$ by $L(f)(x) = y_0 + \int_{x_0}^{x} F(x, f(x))dx$. By suitably topologising X, L is continuous (provided F is nice enough!). A solution of $y' = F(x, y)$ is a fixed-point of L.

Proposition 3.22 *Every connected Tychonoff space having at least two points must have at least $2^{\aleph_0}$ points.*

Proof. Let X be a connected Tychonoff space having at least two points. It suffices to find a surjection $X \to [0, 1]$. Let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Then $\{x_2\}$ is closed, so there is a continuous function $f : X \to [0, 1]$ with $f(x_1) = 0$ and $f(x_2) = 1$. By Theorem 3.20, f is a surjection. ∎

Definition 3.23 *Let X be a space. A* path *in X is a continuous function $\pi : [0, 1] \to X$, where $[0, 1]$ has the usual topology. If $\pi(0) = x$ and $\pi(1) = y$, then the path is* from x to y.

Two points $x, y \in X$ are path connected in X *iff there is a path in X from x to y.*

X is path connected *iff every pair of points of X are path connected in X.*

As in Definition 3.12, the relation "are path connected in X" is an equivalence relation, so gives rise to the idea of the path component *of a point in a space, i.e. the largest path connected subset containing that point.*

Sometimes it is convenient to replace $[0, 1]$ by some other closed interval. By Proposition 3.18 and Theorem 3.19, if two points are path connected in X then they are connected in X. Clearly path connected spaces are connected.

Example 3.24 *(i) connected spaces need not be path connected;*

(ii) path components need not be closed;

(iii) the path analogues of Theorem 3.14 and Corollary 3.15 are not valid.

Consider the two subsets

$$X_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 \ / \ 0 < x_1 < \infty, x_2 = \sin\frac{1}{x_1}\right\} \quad \text{and} \quad X_2 = \{(0, x_2) \in \mathbb{R}^2 \ / \ -1 \le x_2 \le 1\}$$

of $\mathbb{R}^2$. Let $X = X_1 \cup X_2$ and give X the usual topology. Note that in X, $\overline{X_1} = X$. Since X_1 is homeomorphic to $(0, \infty)$ by projection on the first coordinate, by Theorem 3.19, X_1 is connected, so by Corollary 3.15, X is connected. However, it can be shown that X is not path connected. The path components of X are X_1 and X_2. ■

Theorem 3.25 *Let $f : X \to Y$ be continuous and C a path-connected subset of X. Then $f(C)$ is path-connected.*

Proof. If $y_1, y_2 \in f(C)$ then there are $x_1, x_2 \in C$ so that $f(x_i) = y_i$. Let $\pi : [0, 1] \to C$ be a path from x_1 to x_2. Then $f\pi$ is a path from y_1 to y_2. ■

Theorem 3.26 *The topological product of path-connected spaces is path-connected.*

Proof. Let $\{X_\alpha \ / \ \alpha \in A\}$ be a family of path-connected spaces, let $X = \Pi X_\alpha$ and let $x, y \in \Pi X_\alpha$. Then for each $\alpha \in A$, there is a path $p_\alpha : [0, 1] \to X_\alpha$ such that $p_\alpha(0) = x_\alpha$ and $p_\alpha(1) = y_\alpha$. Define $p : [0, 1] \to X$ by $p(t)_\alpha = p_\alpha(t)$. By Proposition 1.26 p is a path from $p(0) = x$ to $p(1) = y$. ■

Definition 3.27 *A space X is* locally connected *at $x \in X$ iff for each neighbourhood U of x, there is a neighbourhood V of x so that every two points of V are connected in U.*

X is locally path-connected *at $x \in X$ iff for each neighbourhood U of x, there is a neighbourhood V of x so that every two points of V are path-connected in U.*

Say that X is locally (path-)connected *iff X is locally (path-)connected at each $x \in X$.*

Lemma 3.28 *A space X is locally (path-)connected at $x \in X$ iff every neighbourhood of x contains a (path-)connected neighbourhood of x.*

Proof. $\Rightarrow$: Let U be a neighbourhood of x. Then there is a neighbourhood V of x so that for each $y \in V$, x and y are (path-)connected in U. For each $y \in V$, let C_y be a connected set containing x and y (a path from x to y) so that $C_y \subset U$. Let $N = \cup_{y \in V} C_y$. Then $V \subset N$, so N is a neighbourhood of x. Also $N \subset U$ and N is (path-)connected.

$\Leftarrow$: trivial. ■

Theorem 3.29 *Let X be a space. The following are equivalent:*

(i) X is locally (path-)connected;

(ii) the (path) components of every open subspace of X are open in X;

(iii) the (path-)connected open subsets of X form a basis for the topology of X.

Proof. (i)$\Rightarrow$(ii): Let U be an open subspace of X. Clearly U is locally (path-) connected by (i), so by Lemma 3.28 the (path) components of U are open in U and hence in X.

(ii)$\Rightarrow$(iii): Let U be any open subspace of X. By (ii), the (path) components of U are open in X. Hence U is the union of a collection of (path-) connected open subsets of X, so such sets form a basis.

(iii)$\Rightarrow$(i): Suppose $x \in X$ and U is an open neighbourhood of x. By (iii), U is the union of a collection of (path-)connected open sets. One of these must contain x, so X is locally (path-)connected. ■

Theorem 3.30 *Let X be a locally path-connected space. Then each path component of X is both open and closed in X.*

Proof. Let C be any path component of X. Then by Theorem 3.29(ii), C is open, hence all path components are open. $X - C$, the union of all path components other than C, is open, so C is closed. ∎

Corollary 3.31 *Every connected, locally path-connected space is path-connected.*

Proof. By Theorems 3.13(v) and 3.30. ∎

Exercises

1. Let X be a second countable space. Show that every family of disjoint open subsets of X is countable.

2. Let X be a second countable space and let $\mathcal{B}$ be any basis for X. Prove that $\mathcal{B}$ contains a countable basis.

3. Let X be a metrisable, separable space. Prove that X is second countable. Can we replace "metrisable" by "first countable" in this statement?

4. Prove that the right half-open interval topology of Example 1.4 is not second countable but is separable. Deduce from Exercise 3 that this topology is not metrisable.

5. Let $l^2 = \left\{\langle x_1, x_2, \ldots\rangle \ / \ x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} x_i^2 < \infty\right\}$. Show that $d : l^2 \times l^2 \to [0, \infty)$ defined by $d(x, y) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{\frac{1}{2}}$, where $x = \langle x_1, x_2, \ldots\rangle, y = \langle y_1, y_2, \ldots\rangle \in l^2$, is a metric on l^2. Prove that l^2 with this metric is separable.

6. Let C be a subset of a topological space. Prove the equivalence of the following three conditions:

   (i) C is disconnected;

   (ii) there are $C_1, C_2 \subset X$ such that $C = C_1 \cup C_2$, $C_1 \cap \overline{C_2} = \varnothing = \overline{C_1} \cap C_2$, $C_1 \neq \varnothing$, $C_2 \neq \varnothing$.

   (iii) there are open sets $U, V \subset X$ such that $C \subset U \cup V$, $C \cap U \cap V = \varnothing$, $C \cap U \neq \varnothing$, $C \cap V \neq \varnothing$.

7. Consider the possibility of replacing $C \cap U \cap V = \varnothing$ in Exercise 6(iii) by $U \cap V = \varnothing$.

   (a) Verify that such a replacement is valid if X is metrisable.

   (b) Give a counterexample illustrating that such a replacement is invalid in a general topological space.

8. Consider the two statements about a topological space X:

   (a) X is connected;

   (b) for each subset Y of X for which $\varnothing \neq Y \neq X$, $\mathrm{fr}Y \neq \varnothing$.

   Decide whether each of the implications (a)$\Rightarrow$(b) and (b)$\Rightarrow$(a) is true or false, giving a complete justification for your answer.

9. The Customs Passage Theorem asserts that for one to bring goods into a country from outside, one must cross the frontier (so customs officials need only guard the frontier). Put this theorem into a topologically precise form using the concepts of this chapter and prove your version of the theorem.

10. Let $\mathbb{R}^\omega$ denote the product of a countable infinity of copies of $\mathbb{R}$ (i.e. $\mathbb{R}^\omega = \Pi_{n\in\omega} X_n$, where $\omega = \{0, 1, 2, \ldots\}$ and $X_n = \mathbb{R}$ for each n). Show that with the box topology, $\mathbb{R}^\omega$ is not connected. (Hint: consider the subset consisting of all bounded sequences).

11. Prove that a space X is connected iff for each open cover $\{U_\alpha \; / \; \alpha \in A\}$ of X and for each $x_1, x_2 \in X$, there is a finite sequence $\alpha_1, \ldots, \alpha_n \in A$ so that $x_1 \in U_{\alpha_1}$, $x_2 \in U_{\alpha_n}$ and $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \varnothing$ for each $i = 1, \ldots, n-1$.

12. Show that $\mathbb{R}$ and $\mathbb{R}^n$ $(n > 1)$ are not homeomorphic.

13. Let X be as in Exercise 1.18. Prove that X is separable but not second countable. Deduce that X is not metrisable.

14. Prove that every separable, locally connected space has only countably many components. Give an example of a separable space having uncountably many components.