# 1 BASIC NOTIONS

**Definition 1.1** *A* topological space *is a pair* $(X, \mathcal{T})$ *(usually abbreviated to* $X$*) where* $X$ *is a set and* $\mathcal{T}$ *a family of subsets of* $X$ *such that:*

1. $\varnothing \in \mathcal{T}$;
2. $X \in \mathcal{T}$;
3. *for each* $U, V \in \mathcal{T}$, $U \cap V \in \mathcal{T}$;
4. *for each* $\mathcal{F} \subset \mathcal{T}$, $\cup\mathcal{F} \in \mathcal{T}$, *where* $\cup\mathcal{F} = \{x \in X \ / \ \text{there is } U \in \mathcal{F} \text{ such that } x \in U\}$.

$\mathcal{T}$ *is called a* topology *for* $X$ *and the elements of* $\mathcal{T}$ *are called* open sets.

**Definition 1.2** *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *be topological spaces and* $f : X \to Y$ *a function. Then* $f$ *is* continuous *iff for each* $U \in \mathcal{U}$, *we have* $f^{-1}(U) \in \mathcal{T}$. *A continuous function is also called a* map. *A continuous bijection whose inverse is also continuous is called a* homeomorphism. *Spaces* $X$ *and* $Y$ *are* homeomorphic *iff there is a homeomorphism from* $X$ *to* $Y$ *(equivalently, from* $Y$ *to* $X$*).*

**Definition 1.3** *A property of topological spaces is called a* topological property *iff whenever it is possessed by a given space it is also possessed by all homeomorphic spaces.*

The fundamental objects of study in topology are the topological spaces and maps: they form a category. The primary goal of topology is to classify topological spaces up to homeomorphism and the principal tool is the topological property. The general task is hopeless, but there has been success in special cases: moreover, the attempt has led to many interesting and valuable results and applications. For example, the study of fixed points is useful in the theory of differential equations; the study of singularities of differentiable maps has resulted in a classification of these in low dimensions, the finiteness of this classification leading to the practicality of catastrophe theory with its applications in diverse areas such as Physics, Biology, Economics, Linguistics; the study of the placement of sets, especially of curves, in 3-dimensional space has led to knot theory with applications in Biology and Physics.

Any attempt to classify in some suitable way the finite topological spaces is likely to be unsuccessful, and as the following table shows, even the number of topologies on a finite set follows a strange pattern; indeed there seems to be no simple formula for this number. In this table, the number of distinct topologies on a set of $n$ elements is denoted by $t(n)$ and the number of homeomorphism classes of such topologies is denoted by $h(n)$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $t(n)$ | 1 | 4 | 29 | 355 | 6942 | 209527 | 9535241 | 642779354 |
| $h(n)$ | 1 | 3 | 9 | 33 | 139 | 718 | 4535 | |

**Example 1.4** *Examples of topological spaces.*

1. Declare any subset of $\mathbb{R}$ to be open iff it is a union of open intervals. Then the set of open subsets of $\mathbb{R}$ forms a topology on $\mathbb{R}$, called the *usual topology*. More generally, declare any subset of $\mathbb{R}^n$ to be open iff it is a union of open "cubes", i.e. sets of the form $\{(x_1, x_2, \ldots, x_n) \ / \ a_i < x_i < b_i\}$, for some $a_i$ and $b_i$. Again the family of open sets forms a topology on $\mathbb{R}^n$ called the *usual topology*.

2. Let $(X, d)$ be any metric space. Declare a subset $U \subset X$ open iff for each $x \in U$ there is $r > 0$ such that $B_d(x; r) \subset U$, where $B_d(x; r)$ consists of all $y \in X$ for which $d(x, y) < r$. This is the *topology induced by the metric* $d$. It is the same as the usual topology on $\mathbb{R}$ and $\mathbb{R}^n$ if we take as $d$ the Pythagorean metric.

   If $(X, \mathcal{T})$ is a topological space such that there is a metric $d$ on $X$ for which $\mathcal{T}$ is induced by $d$ then we say that $(X, \mathcal{T})$ is *metrisable*. A big sub-problem of topology is to decide when a given space is metrisable.

3. Let $X$ be any set and topologise $X$ by declaring only $\varnothing$ and $X$ to be open. This, the smallest possible topology on $X$, is called the *indiscrete topology*.

4. Topologise any set $X$ by declaring all subsets to be open. This, the largest topology on $X$, is called the *discrete topology*. Unlike the indiscrete topology, it is a special case of a metric topology, where the metric is the discrete metric in which the distance between two different points is always 1.

5. Declare any subset of $\mathbb{R}$ to be open iff it is a union of intervals of the form $[a, b)$. This, the *right half-open interval topology*, is not metrisable, but obviously it differs from the indiscrete topology.

**Example 1.5** *Examples of continuous functions.*

1. Any function whose range is indiscrete is continuous.

2. Any function whose domain is discrete is continuous.

3. Any function between metric spaces is continuous in the metric sense iff it is continuous in the sense of Definition 1.2 when the spaces are given the topologies induced by the metrics.

**Definition 1.6** *Let $(X, \mathcal{T})$ be a topological space. A subfamily $\mathcal{B}$ of $\mathcal{T}$ is a* basis *for $\mathcal{T}$ provided every member of $\mathcal{T}$ is a union of members of $\mathcal{B}$. [Thus $\{[a, b) \ / \ a, b \in \mathbb{R}\}$ is a basis for the right half-open interval topology of 5 of Example 1.4.] A subfamily $\mathcal{S}$ of $\mathcal{T}$ is a* sub-basis *provided the family of all finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{T}$. Any family $\mathcal{F}$ of subsets of $X$ is a sub-basis for a unique topology on $X$, called the* topology generated by *$\mathcal{F}$.*

**Proposition 1.7** *A family $\mathcal{B}$ of subsets of a set $X$ is a basis for a topology on $X$ if and only if:*

1. *For each $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$; and*

2. *For each $B_1, B_2 \in \mathcal{B}$ and each $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ so that $x \in B_3 \subset B_1 \cap B_2$.*

Proof. $\Rightarrow$: Suppose $\mathcal{B}$ is a basis for a topology $\mathcal{T}$. By definition, $X$ is open so is a union of members of $\mathcal{B}$; this is equivalent to 1. Given $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, it follows that $B_1, B_2 \in \mathcal{T}$, so that $B_1 \cap B_2 \in \mathcal{T}$. Thus $B_1 \cap B_2$ is a union of members of $\mathcal{B}$, one of which must contain $x$ and lie in $B_1 \cap B_2$.

$\Leftarrow$: Given a collection $\mathcal{B}$ satisfying 1 and 2, let $\mathcal{T}$ denote the family of all subsets of $X$ which are a union of some members of $\mathcal{B}$.

1. $\varnothing \in \mathcal{T}$ since $\varnothing$ is the union of the subcollection $\varnothing$ of $\mathcal{B}$.

2. $X \in \mathcal{T}$ by condition 1 on $\mathcal{B}$.

3. If $U, V \in \mathcal{T}$ and $x \in U \cap V$ then there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U$ and $x \in B_2 \subset V$. By condition 2 on $\mathcal{B}$ there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$; thus $x \in B_3 \subset U \cap V$. This shows that for each $x \in U \cap V$ there is $B_x \in \mathcal{B}$ such that $x \in B_x \subset U \cap V$, hence that $U \cap V$ is a union of members of $\mathcal{B}$ and so $U \subset V \in \mathcal{T}$.

4. For each $\mathcal{F} \subset \mathcal{T}$, $\cup\mathcal{F} \subset \mathcal{T}$ is obvious. ■

Often it is more convenient to specify a topology by describing a basis for the topology. The criterion of Proposition 1.7 is useful in this regard.

**Definition 1.8** *Let $(X, \mathcal{T})$ be a topological space. A subset $N$ of $X$ is a* neighbourhood *of $x \in X$ iff there is $U \in \mathcal{T}$ such that $x \in U \subset N$. Let $\mathcal{N}(x)$ denote the collection of all neighbourhoods of $x$. A subfamily $\mathcal{M}$ of $\mathcal{N}(x)$ is a* basis of neighbourhoods *at $x$ provided for each $N \in \mathcal{N}(x)$ there is $M \in \mathcal{M}$ such that $M \subset N$.*

Note that there is no requirement that neighbourhoods be open. Occasionally authors do insist that their neighbourhoods are open.

**Proposition 1.9** *Let $(X, \mathcal{T})$ and $\mathcal{N}(x)$ be as above. Then*

1. *$N \in \mathcal{N}(x) \Rightarrow x \in N$;*

2. *$N_1 \in \mathcal{N}(x)$ and $N_1 \subset N_2 \subset X \Rightarrow N_2 \in \mathcal{N}(x)$;*

3. *$N_1, N_2 \in \mathcal{N}(x) \Rightarrow N_1 \cap N_2 \in \mathcal{N}(x)$;*

4. *For each $N \in \mathcal{N}(x)$ there is $M \in \mathcal{N}(x)$ such that $y \in M \Rightarrow N \in \mathcal{N}(y)$.*

Proof. Easy. ■

**Lemma 1.10** *Let $(X, \mathcal{T})$ and $\mathcal{N}(x)$ be as above and let $U \subset X$. Then $U \in \mathcal{T}$ iff whenever $x \in U$, it follows that $U \in \mathcal{N}(x)$.*

Proof. $\Rightarrow$: obvious.
$\Leftarrow$: For each $x \in U$ there is $U_x \in \mathcal{T}$ such that $x \in U_x \subset U$. By Definition 1.1, $\cup_{x \in U} U_x \in \mathcal{T}$. But $U = \cup_{x \in U} U_x$ , so $U \in \mathcal{T}$. ■

**Proposition 1.11** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set $X$ with $\mathcal{N}_i(x)$ the corresponding families of neighbourhoods. Then $\mathcal{T}_1 = \mathcal{T}_2$ iff for each $x \in X$, $\mathcal{N}_1(x) = \mathcal{N}_2(x)$.*

Proof. $\Rightarrow$: trivial.
$\Leftarrow$: Let $U \subset X$. By Lemma 1.10 and hypothesis, $U \in \mathcal{T}_1$ iff for each $x \in U$, $U \in \mathcal{N}_1(x)$ iff for each $x \in U$, $U \in \mathcal{N}_2(x)$ iff $U \in \mathcal{T}_2$, so $\mathcal{T}_1 = \mathcal{T}_2$ as required. ■

**Theorem 1.12** *Let $X$ be a set and suppose that to each $x \in X$, there is assigned a non-empty family $\mathcal{N}(x)$ of subsets of $X$ satisfying 1 to 4 of Proposition 1.9. Then there is a unique topology on $X$ having $\mathcal{N}(x)$ as its family of neighbourhoods for each $x \in X$.*

Proof. Let $\mathcal{T} = \{U \subset X \ / \ x \in U \Rightarrow U \in \mathcal{N}(x)\}$. Then $\mathcal{T}$ is clearly a topology for $X$. For each $x \in X$, let $\mathcal{M}(x)$ denote the family of neighbourhoods of $x$ in $(X, \mathcal{T})$. We show that $\mathcal{M}(x) = \mathcal{N}(x)$.

Suppose $M \in \mathcal{M}(x)$. Then there is $U \in \mathcal{T}$ such that $x \in U \subset \mathcal{M}$. By definition of $\mathcal{T}$, $x \in U \Rightarrow U \in \mathcal{N}(x)$, so by 1 of Proposition 1.9, $M \in \mathcal{N}(x)$. Thus $\mathcal{M}(x) \subset \mathcal{N}(x)$.

Suppose $N \in \mathcal{N}(x)$. Let $U = \{y \in X \ / \ N \in \mathcal{N}(y)\}$. If $y \in U$, then by 4 of Proposition 1.9 there is $M_y \in \mathcal{N}(y)$ such that $z \in M_y \Rightarrow N \in \mathcal{N}(z)$. By definition of $U$, we have $M_y \subset U$, so, since $M_y \in \mathcal{N}(y)$, we also have $U \in \mathcal{N}(y)$, so by definition of $\mathcal{T}$, $U \in \mathcal{T}$. Now by Lemma 1.10, $U \in \mathcal{M}(x)$, so, since $U \subset N$, $N \in \mathcal{M}(x)$ by 2 of Proposition 1.9. Hence $\mathcal{N}(x) \subset \mathcal{M}(x)$, and so $\mathcal{M}(x) = \mathcal{N}(x)$ as required. ∎

The significance of Theorem 1.12 is that we could have defined a topological space by axiomatising its neighbourhood systems using conditions 1 to 4 of Proposition 1.9. Often this approach is more convenient.

**Definition 1.13** *A subset $A$ of a topological space $(X, \mathcal{T})$ is* closed *iff $(X - A) \in \mathcal{T}$.*

**Proposition 1.14** *Closed sets satisfy the following:*

1. *$X$ is closed;*

2. *$\varnothing$ is closed;*

3. *the union of any two closed sets is closed;*

4. *the intersection of an arbitrary family of closed sets is closed.*

Proof. Easy. ∎

Compare conditions 1 to 4 above with those of Definition 1.1 for open sets.

**Definition 1.15** *Let $A \subset X$. By 4 of Proposition 1.14, the intersection of the (non-empty!) family $\{C \subset X \ / \ A \subset C$ and $C$ is closed$\}$ is a closed set containing $A$: in fact the smallest closed set containing $A$. This set is called the* closure *of $A$, denoted $\bar{A}$ or cl$A$. Dually, using 4 of Definition 1.1, there is a largest open set contained in $A$, called the* interior *of $A$, denoted $\mathring{A}$ or int$A$.*

Of course, we cannot expect the intersection of all open sets containing $A$ to be open nor the union of all closed sets contained in $A$ to be closed. For example, let $A = [0, 1)$, a subset of $\mathbb{R}$ with the usual topology. Then $A$ is neither open nor closed although it is both the intersection of all open sets containing $A$ and the union of all closed sets contained in $A$.

**Definition 1.16** *Let $A \subset X$. A point $x$ of $X$ is an* accumulation point *of $A$ iff $x \in \overline{A - \{x\}}$. The* derived set*, $A'$, of $A$ is the set of all accumulation points. The* frontier*, fr$A$, of $A$ is the set $\bar{A} \cap \overline{X - A}$.*

**Proposition 1.17** *Let $A \subset X$. Then*

$$\bar{A} = \{x \in X \ / \ \textit{for each } N \in \mathcal{N}(x), A \cap N \neq \varnothing\} = A \cup A' = A \cup \mathrm{fr}A.$$

Proof. If $x \notin \bar{A}$,then by Lemma 1.10, $X - \bar{A} \in \mathcal{N}(x)$. Since $A \cap (X - \bar{A}) \subset \bar{A} \cap (X - \bar{A}) = \varnothing$, we deduce that $\{x \in X \ / \ \text{for each } N \in \mathcal{N}(x), A \cap N \neq \varnothing\} \subset \bar{A}$.

Suppose $x \in X$ and $N \in \mathcal{N}(x)$ are such that $A \cap N = \varnothing$. Then there is open $U$ such that $x \in U \subset N \subset X - A$. Thus $A$ is contained in the closed set $X - U$, so $\bar{A} \subset X - U$ and hence $x \notin \bar{A}$. Thus $\bar{A} \subset \{x \in X \ / \ \text{for each } N \in \mathcal{N}(x), A \cap N \neq \varnothing\}$.

If $x \in \bar{A} - A$, then $\overline{A - \{x\}} = \bar{A}$, so $x \in \overline{A - \{x\}}$, i.e. $x \in A'$. Thus $\bar{A} \subset A \cap A'$. Clearly$A \subset \bar{A}$ and if $x \in A'$, then $x \in \overline{A - \{x\}} \subset \bar{A}$, so $A' \subset \bar{A}$.

By definition, $A \cup \mathrm{fr}A \subset \bar{A}$. If $x \in \bar{A} - A$, then $x \in X - A$, so $x \in \overline{X - A}$ and hence $x \in \mathrm{fr}A$: thus $\bar{A} \subset A \cup \mathrm{fr}A$. ∎

**Definition 1.18** *Let $f : X \to Y$ be a function between topological spaces. Then $f$ is* continuous *at $x \in X$ iff for each neighbourhood $N$ of $f(x)$ in $Y$, $f^{-1}(N)$ is a neighbourhood of $x$ in $X$.*

**Theorem 1.19** *Let $f : X \to Y$ be a function between topological spaces. Then the following conditions are equivalent:*

*(i) $f$ is continuous;*

*(ii) for each $x \in X$, $f$ is continuous at $x$;*

*(iii) if $\mathcal{B}$ is a basis for $Y$ then for each $U \in \mathcal{B}$, $f^{-1}(U)$ is open in $X$;*

*(iv) if $\mathcal{S}$ is a sub-basis for $Y$ then for each $U \in \mathcal{S}$, $f^{-1}(U)$ is open in $X$;*

*(v) for each closed subset $C$ of $Y$, $f^{-1}(C)$ is closed in $X$;*

*(vi) for each $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$;*

*(vii) for each $B \subset Y$, $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$;*

*(viii) for each $B \subset Y$, $f^{-1}(\mathring{B}) \subset \operatorname{int}(f^{-1}(B))$.*

Proof. (i)⇒(ii): let $x \in X$ and $N \in \mathcal{N}(f(x))$. Then by definition there is $U$, open in $Y$, such that $f(x) \in U \subset N$. By (i), $f^{-1}(U)$ is open in $X$ and contains $x$, so $f^{-1}(U) \in \mathcal{N}(x)$. Since $f^{-1}(U) \subset f^{-1}(N)$, 2 of Proposition 1.9 tells us that $f^{-1}(N) \in \mathcal{N}(x)$.

(ii)⇒(iii): let $\mathcal{B}$ be a basis for the topology of $Y$ and let $U \in \mathcal{B}$. By Lemma 1.10, it suffices to show that $f^{-1}(U)$ is a neighbourhood of each of its points. But if $x \in f^{-1}(U)$ then $f(x) \in U$, so $U \in \mathcal{N}(f(x))$. Hence by (ii), $f^{-1}(U) \in \mathcal{N}(x)$ as required.

(iii)⇒(iv): trivial.

(iv)⇒(v): suppose $C$ is closed in $Y$ and let $x \in X - f^{-1}(C)$. Then $f(x) \in Y - C$ which is open, so there are $U_1, \ldots, U_n \in \mathcal{S}$ such that $f(x) \in U_1 \cap \ldots \cap U_n \subset Y - C$. By (iv), $f^{-1}(U_i)$ is open, so $f^{-1}(U_1) \cap \ldots \cap f^{-1}(U_n)$ is open in $X$. But $f^{-1}(U_1) \cap \ldots \cap f^{-1}(U_n) = f^{-1}(U_1 \cap \ldots \cap U_n)$, and $x \in f^{-1}(U_1 \cap \ldots \cap U_n) \subset X - f^{-1}(C)$, so $X - f^{-1}(C) \in \mathcal{N}(x)$. Hence by Lemma 1.10, $X - f^{-1}(C)$ is open and hence $f^{-1}(C)$ is closed in $X$.

(v)⇒(vi): given $A \subset X$, $\overline{f(A)}$ is closed in $Y$ so by (v), $f^{-1}(\overline{f(A)})$ is closed in $X$. But $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$, so $\bar{A} \subset f^{-1}(\overline{f(A)})$ and hence $f(\bar{A}) \subset \overline{f(A)}$.

(vi)⇒(vii): given $B \subset Y$, letting $A = f^{-1}(B)$ in (vi), we have $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \bar{B}$, so $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.

(vii)⇒(viii): given $B \subset Y$, apply (vii) to $Y - B$ to obtain $f^{-1}(\mathring{B}) = f^{-1}(Y - \overline{Y - B}) = X - f^{-1}(\overline{Y - B}) \subset X - \overline{f^{-1}(Y - B)} = X - \overline{X - f^{-1}(B)} = \operatorname{int}(f^{-1}(B))$.

(viii)⇒(i): let $U$ be open in $Y$: thus $\mathring{U} = U$, so by (viii), $f^{-1}(U) \subset \operatorname{int}(f^{-1}(U))$. Clearly $\operatorname{int}(f^{-1}(U)) \subset f^{-1}(U)$, so $f^{-1}(U) = \operatorname{int}(f^{-1}(U))$ and hence is open. ∎

**Proposition 1.20** *Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $gf : X \to Z$ is continuous.*

Proof. Obvious from the definition. ∎

Let $A$ be a subset of the space $(X, \mathcal{T})$. By declaring open all subsets of $A$ of the form $U \cap A$ for $U \in \mathcal{T}$ we obtain a topology for $A$ with which $A$ is called a *subspace* of $X$. Let $i : A \to X$ denote the inclusion function.

**Proposition 1.21** *$i : A \to X$ is continuous. Hence if $f : X \to Y$ is continuous then so is $f|A : A \to Y$.*

Proof. If $U$ is open in $X$ then $i^{-1}(U) = U \cap A$ is open in $A$, so $i$ is continuous. The second part follows from Proposition 1.20 since $f|A = fi$. ∎

**Theorem 1.22** *Suppose $\{X_\alpha \;/\; \alpha \in A\}$ is a collection of subspaces of a topological space $X$ whose union is $X$ and $f : X \to Y$ is a function. Let $f_\alpha = f|X_\alpha$ and suppose $f_\alpha$ each is continuous. Then $f$ is continuous provided either:*

*(i) each $X_\alpha$ is open; or*

*(ii) each $X_\alpha$ is closed and $A$ is finite.*

Proof. (i) Let $U$ be open in $Y$. Then for each $\alpha \in A$, $f_\alpha^{-1}(U)$ is open in $X_\alpha$ and hence, since each $X_\alpha$ is open in $X$, $f_\alpha^{-1}(U)$ is open in $X$. But $f^{-1}(U) = \cup_{\alpha \in A} f_\alpha^{-1}(U)$, so $f^{-1}(U)$ is open and $f$ is continuous.

(ii) Let $C$ be closed in $Y$. Then for each $\alpha \in A$, $f_\alpha^{-1}(C)$ is closed in $X_\alpha$ and hence, since each $X_\alpha$ is closed in $X$, $f_\alpha^{-1}(C)$ is closed in $X$. But $f^{-1}(C) = \cup_{\alpha \in A} f_\alpha^{-1}(C)$, a finite union, so $f^{-1}(C)$ is closed and $f$ is continuous. ∎

Note that we cannot in general remove the finiteness condition from (ii). For example, let $X = \mathbb{R}$ and for each $\alpha \in \mathbb{R}$, let $X_\alpha = \{\alpha\}$. For any function $f : \mathbb{R} \to \mathbb{R}$, $f_\alpha$ is continuous, but $f$ need not be continuous.

**Definition 1.23** *Let $\{X_\alpha \;/\; \alpha \in A\}$ be a family of topological spaces and suppose that for each $\alpha, \beta \in A$ with $\alpha \neq \beta$ we have $X_\alpha \cap X_\beta = \varnothing$. By the* topological sum*, $\Sigma X_\alpha$, of these spaces we mean the set $X = \cup_{\alpha \in A} X_\alpha$ topologised by declaring $U \subset X$ open iff for each $\alpha \in A$, $U \cap X_\alpha$ is open in $X_\alpha$.*

Note that the inclusion functions $X_\alpha \to X$ are continuous: in fact $X_\alpha$ is a subspace of $X$.

**Definition 1.24** *Let $\{X_\alpha \;/\; \alpha \in A\}$ be a family of topological spaces. By the* topological product *of this family we mean the set*

$$X = \Pi X_\alpha = \{x : A \to \cup_{\alpha \in A} X_\alpha \;/\; \textit{for each } \alpha \in A, x(\alpha) \in X_\alpha\},$$

*furnished with the* Tychonoff topology*. The family*

$$\{\Pi U_\alpha \;/\; U_\alpha \textit{ is open in } X_\alpha \textit{ and } U_\alpha \neq X_\alpha \textit{ for at most one index } \alpha \in A\}$$

*forms a sub-basis for the Tychonoff topology, and is called the* standard sub-basis*.*

*For each $x \in X$ and $\alpha \in A$ , denote $x(\alpha)$ by $x_\alpha$. Define for each $\alpha \in A$, the $\alpha$th projection $\pi_\alpha : X \to X_\alpha$ by $\pi_\alpha(x) = x_\alpha$.*

$$\{\Pi U_\alpha \;/\; U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha \neq X_\alpha \text{ for at most finitely many indices } \alpha \in A\}$$

is called the *standard basis* for the Tychonoff topology.

If $A = \{1, \ldots, n\}$ then $\Pi X_\alpha$ is usually written $X_1 \times \ldots \times X_n$.

**Proposition 1.25** *The Tychonoff topology is the smallest topology on $X$ for which the projections $\pi_\alpha$ are continuous.*

Proof. $\pi_\alpha$ is continuous when $X$ is given the Tychonoff topology, for if $U_\alpha$ is open in $X_\alpha$ then $\pi_\alpha^{-1}(U_\alpha) = \Pi_{\beta \in A} U_\beta$, where $U_\beta = X_\beta$ if $\beta \neq \alpha$. By definition, $\Pi_{\beta \in A} U_\beta$ is open, so $\pi_\alpha$ is continuous.

Let $\mathcal{T}$ be any topology on $X$ with respect to which each projection is continuous. We must show that $\mathcal{T}$ contains the Tychonoff topology. It suffices to show that $\mathcal{T}$ contains the standard sub-basis. Given $\{U_\alpha\}$ with each $U_\alpha$ open in $X_\alpha$ and $U_\alpha = X_\alpha$ except possibly for the one index $\beta$. Then $\Pi U_\alpha = \pi_\beta^{-1}(U_\beta)$. By continuity of $\pi_\beta$, the set $\pi_\beta^{-1}(U_\beta)$ is open in $\mathcal{T}$. Thus $\Pi U_\alpha \in \mathcal{T}$. ∎

**Proposition 1.26** *A function $f : Y \to \Pi X_\alpha$, where $Y$ and each $X_\alpha$ are topological spaces, is continuous iff for each $\alpha \in A$, $\pi_\alpha f$ is continuous.*

Proof. We need only show that if $\pi_\alpha f$ is continuous for each $\alpha$, then $f$ is continuous.

Let $\Pi U_\beta$ be a member of the standard sub-basis of $X$, with $U_\beta = X_\beta$ unless $\beta = \alpha$. Now $\pi_\alpha(\Pi U_\beta) = U_\alpha$ which is open in $X_\alpha$ so by continuity of $\pi_\alpha f$, $(\pi_\alpha f)^{-1}(U_\alpha)$ is open. Of course, $(\pi_\alpha f)^{-1}(U_\alpha) = f^{-1}(\Pi U_\beta)$. ∎

**Corollary 1.27** *Let $\{f_\alpha : X_\alpha \to Y_\alpha \ / \ \alpha \in A\}$ be a family of continuous functions. Define the product of this family, $f : X \to Y$, by $f(x)(\alpha) = f_\alpha(x_\alpha)$. Then $f$ is continuous.*

Proof. Let $\rho_\alpha : Y \to Y_\alpha$ be the $\alpha$th projection. Then we have $\rho_\alpha f = f_\alpha \pi_\alpha$. Since $f_\alpha \pi_\alpha$ is continuous, so is $f$ by Proposition 1.26. ∎

**Proposition 1.28** *For each $\alpha \in A$, $\pi_\alpha : X \to X_\alpha$ is an* open map*, i.e., for each open $U \subset X$, $\pi_\alpha(U)$ is open in $X_\alpha$.*

Proof. Given $\alpha \in A$ and $U$ open in $X$, suppose that $x_\alpha \in \pi_\alpha(U)$. Then there is $x \in U$ such that $\pi_\alpha(x) = x_\alpha$. Thus there are $\alpha_1, \ldots, \alpha_n \in A$ and open subsets $U_{\alpha_i}$ of $X_{\alpha_i}$ such that $x \in \Pi U_\beta \subset U$, where $U_\beta = X_\beta$ if $\beta \notin \{\alpha_1, \ldots, \alpha_n\}$. Then $x_\alpha \in U_\alpha = \pi_\alpha(\Pi U_\beta) \subset \pi_\alpha(U)$, so $\pi_\alpha(U)$ is open. ∎

We cannot in general replace "open" by "closed" in Proposition 1.28, i.e. $\pi_\alpha$ is not in general a *closed* map. For example, $\mathbb{R}^2$ is (homeomorphic to) the product of $\mathbb{R}$ with itself but projection onto either factor of the closed set $\{(x, y) \in \mathbb{R}^2 \ / \ xy = 1\}$ is the non-closed set $(-\infty, 0) \cup (0, \infty)$.

**Definition 1.29** *Let $X$ be any topological space and $\sim$ an equivalence relation on $X$. Let $X/\sim$ denote the set of equivalence classes and $\pi : X \to X/\sim$ the natural projection. By the* quotient space *of $X$ by $\sim$ is meant the set $X/\sim$ topologised by declaring $U \subset X/\sim$ to be open iff $\pi^{-1}(U)$ is open.*

**Proposition 1.30** *Let $X$ and $Y$ be spaces, $\sim$ an equivalence relation on $X$ and $f : X/\sim \to Y$ a function. Then $f$ is continuous iff $f\pi$ is continuous.*

Proof. Easy. ∎

**Definition 1.31** *Let $X$ be any space. By the* cone *on $X$ is meant the space $cX$ defined as follows: let $v$ be any point not in $X \times [0, 1]$. Give $\{v\}$ the only possible topology and $X \times [0, 1]$ the Tychonoff topology. Define the equivalence relation $\sim$ on the topological sum $\{v\} + (X \times [0, 1])$ to be that generated by $v \sim (x, 1)$, for each $x \in X$. Then $cX$ is the quotient $[\{v\} + (X \times [0, 1])]/\sim$. The equivalence class of $v$ is the* vertex *of the cone. We can think of $X$ as a subspace of $cX$ by identifying $x \in X$ with the equivalence class of $(x, 0)$.*

**Definition 1.32** *Let $f : X \to Y$ be a map. By the* mapping cylinder *of $f$ is meant the space $Z_f$ defined as follows: consider the topological sum $(X \times [0,1]) + Y$ (if $(X \times [0,1]) \cap Y \neq \varnothing$, replace $Y$ by a homeomorph which is disjoint from $X \times [0,1]$ ). Define the equivalence relation $\sim$ on $(X \times [0,1]) + Y$ to be that generated by $(x,1) \sim f(x)$. Then $Z_f$ is the quotient space $[(X \times [0,1]) + Y]/\sim$. As in the case of $cX$, we can think of $X$ as a subspace of $Z_f$. $Y$ is also able to be treated as a subspace of $Z_f$.*

## Exercises

1. Let $X$ be a set and $\mathcal{T}$ a family of subsets of $X$. Prove that $\mathcal{T}$ is a topology for $X$ iff
   (a) the intersection of any finite sub-family of $\mathcal{T}$ is a member of $\mathcal{T}$ and
   (b) the union of any sub-family of $\mathcal{T}$ is a member of $\mathcal{T}$.

2. Construct all possible topologies on a set of 3 elements and divide them into homeomorphism classes.

3. Let $X$ be a set. Let $\mathcal{T}$ be the family of subsets of $X$ consisting of $\varnothing$ together with all subsets whose complement in $X$ is finite. Verify that $\mathcal{T}$ is a topology on $X$. $\mathcal{T}$ is called the *cofinite* topology. Note that we can replace "finite" by "countable" to get the *cocountable* topology.

4. Construct a basis for the topology in Exercise 3 which differs from the topology itself.

5. Show that the countable family $\mathcal{B} = \{(a,b) \ / \ a < b$ and $a$ and $b$ are rational$\}$ is a basis for the usual topology on $\mathbb{R}$.

6. Show that the countable family $\mathcal{B} = \{[a,b) \ / \ a < b$ and $a$ and $b$ are rational$\}$ is a basis for a topology on $\mathbb{R}$. Is this the right half-open interval topology of Example 1.4?

7. The subsets $A_\pm, \ldots, I_\pm, X$ and $Y$ of $\mathbb{R}^3$ are defined by

   $A_+ = [3,5] \times [-1,1] \times [-3,7]$,
   $B_+ = [-5,-3] \times [-1,1] \times [-3,7]$,
   $C_+ = [-5,5] \times [-1,1] \times [5,7]$,
   $D_+ = [-5,5] \times [-1,1] \times [-3,-1]$,
   $E_+ = [-1,1] \times [-1,1] \times [5,11]$,
   $F_+ = [-1,9] \times [-1,1] \times [9,11]$,
   $G_+ = [7,9] \times [-1,1] \times [-4,11]$,
   $H_+ = A_+ \cup \ldots \cup G_+$,
   $I_+ = T(H_+)$,
   $X = H_+ \cup H_-$,

   $A_- = [-1,1] \times [3,5] \times [-7,3]$,
   $B_- = [-1,1] \times [-5,-3] \times [-7,3]$,
   $C_- = [-1,1] \times [-5,5] \times [-7,-5]$,
   $D_- = [-1,1] \times [-5,5] \times [1,3]$,
   $E_- = [-1,1] \times [-1,1] \times [-11,-5]$,
   $F_- = [-1,9] \times [-1,1] \times [-11,-9]$,
   $G_- = [7,9] \times [-1,1] \times [-11,4]$,
   $H_- = A_- \cup \ldots \cup G_-$,
   $I_- = T^{-1}(H_-)$,
   $Y = I_+ \cup I_-$.

   In the definition above, $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x,y,z+4)$.
   Prove that
   (a) $X$ is homeomorphic to $Y$;
   (b) there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(X) = Y$.

8. Let $X$ be a topological space. Define two functions c,cl: $\mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$, the *power set* of $X$, consists of all subsets of $X$, by $\mathrm{c}(A) = X - A$ and $\mathrm{cl}(A) = \bar{A}$.

   Prove that there are at most 14 distinct subsets of $X$ obtained from a given subset $A$ by successive applications of the functions c and cl. Give an example of a subset of $\mathbb{R}$ for which there are 14 such distinct sets.

9. Is $\mathring{A} = A - \text{fr}A$?

10. Say that a point $x$ is *near* a subset $A$ of a topological space $X$ and write $x\nu A$ iff $x \in \bar{A}$. Let $f : X \to Y$ be a function where $X$ and $Y$ are two topological spaces. Let $x \in X$. Prove that $f$ is continuous at $x$ iff for each $A \subset X$, $x\nu A \Rightarrow f(x)\nu f(A)$.

11. Let $X$ and $Y$ be two finite sets. It is desired to find all triples $(f, g, h)$ of functions from $X$ to $Y$ such that whatever topologies are imposed on $X$ and $Y$, if $f$ and $g$ are continuous then so is $h$. One way to find these is to use a computer to search for such triples. However, the number of topologies on a set of $n$ elements grows rather quickly with $n$ so one must look for ways of shortening the procedure. Prove that $(f, g, h)$ is such a triple iff for each $V \subset Y$, $h^{-1}(V)$ is one of the following: $\varnothing, X, f^{-1}(V), g^{-1}(V), f^{-1}(V) \cap g^{-1}(V)$, and $f^{-1}(V) \cup g^{-1}(V)$.

12. Cantor's ternary set, $C$, is defined to be the intersection of the sequence $C_0, C_1, \ldots$, where $C_0 = [0, 1]$ and for $i > 0$, $C_i$ is the union of $2^i$ closed intervals obtained from the $2^{i-1}$ closed intervals of $C_{i-1}$ by removing from each of these intervals the open middle third of the interval. Thus

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \text{ etc.}$$

Prove that $C$ is homeomorphic to $\mathbf{2}^{\mathbb{N}}$ where $\mathbf{2}$ is the discrete space on two points, and $\mathbf{2}^{\mathbb{N}}$ denotes the product of countably many copies of the space $\mathbf{2}$ (i.e., each of the factor spaces is the space $\mathbf{2}$).

13. Let $\{X_\alpha \;/\; \alpha \in A\}$ be a family of topological spaces. Show that the family $\{\Pi U_\alpha \;/\; U_\alpha \text{ is open in } X_\alpha\}$ forms a basis for a topology on $\Pi X_\alpha$. This topology is called the *box topology* and seems to be a more natural generalisation of the topology on $\mathbb{R}^2$ having as basis the open rectangles. Show that for $A$ finite, the Tychonoff topology and the box topology are the same, but that they can differ for $A$ infinite. In the latter case, the box topology has too many open sets. Construct an example showing that Proposition 1.26 is false if we use the box topology in place of the Tychonoff topology.

14. Show that the cone $cX$ on $X$ is really just a particular example of a mapping cylinder.

15. Let $X = [-1, 1]$ and $Y = \{(x, y) \in \mathbb{R}^2 \;/\; 0 \le y \le 1 \text{ and } |x| \le 1 - y\}$, with topologies inherited as subspaces of $\mathbb{R}$ and $\mathbb{R}^2$ respectively. (Thus $Y$ consists of the union of all line segments joining $(0, 1)$ to the closed line segment from -1 to 1 on the $x$-axis.) Prove that $cX$ is homeomorphic to $Y$. What happens if we replace $X$ by the open interval and let $Y$ join $(0, 1)$ to the open line segment?

16. Let $X = \{(x, y, z) \in \mathbb{R}^3 \;/\; x^2 + y^2 = 1 \text{ and } 0 < z < 1\}$, $Y = [-1, 1] \times (0, 1)$ and $Z = (-1, 1) \times (0, 1) \cup \{2\} \times (0, 1)$. Give $X$ the subspace topology inherited from the usual topology on $\mathbb{R}^3$, and $Y$ the subspace topology inherited from the usual topology on $\mathbb{R}^2$.

    Define $\sim$ on $Y$ by declaring $(x, y) \sim (x', y')$ if and only if either $(x, y) = (x', y')$ or $\{x, x'\} = \{-1, 1\}$ and $y = y'$.

    For each $a, b, r \in (0, 1)$ with $a < b$, let $R_{a,b,r} = ((-1, -r) \cup (r, 1) \cup \{2\}) \times (a, b)$, and let

$$\begin{aligned} \mathcal{B} &= \{U \cap (-1, 1) \times (0, 1) \;/\; U \text{ is open in the usual topology on } \mathbb{R}^2\} \\ &\quad \cup \{R_{a,b,r} \;/\; a, b, r \in (0, 1) \text{ and } a < b\}. \end{aligned}$$

(a) Prove that $\sim$ is an equivalence relation on $Y$.

(b) Construct a natural homeomorphism $X \to Y/\sim$.

(c) Prove that $\mathcal{B}$ is a basis for a topology on $Z$.

(d) Construct a natural homeomorphism $Y/\sim \to Z$ when $Z$ has the topology considered in (c).

17. Let $X = X_1 \cup \{0\} \times \{0\} \times \mathbb{R}$, where $X_1 = \{(x, y) \in \mathbb{R}^2 \ / \ x > 0\}$,
$Y = \{(x, y) \ / \ 0 < x < 1 \text{ and } -x < y < x\} \cup \{0\} \times \{0\} \times (-1, 1)$ and
$Z = [0, 1) \times (-1, 1)$.

Give $Z$ the subspace topology inherited from the usual topology on $\mathbb{R}^2$.

For each $a \in \mathbb{R}$ and $r > 0$, let

$$N_{a,r} = \{(x, y) \in \mathbb{R}^2 \ / \ 0 < x < r \text{ and } (a - r)x < y < (a + r)x\} \cup \{0\} \times \{0\} \times (a - r, a + r).$$

Let

$$\mathcal{B} = \{U \cap X_1 \ / \ U \text{ is open in the usual topology on } \mathbb{R}^2\} \cup \{N_{a,r} \ / \ a \in \mathbb{R} \text{ and } r > 0\}.$$

(a) Draw pictures of $N_{a,r}$ for two different values of $(a, r)$.

(b) Prove that $\mathcal{B}$ is a basis for a topology on $X$.

(c) Construct a natural homeomorphism $Y \to Z$ when $Y$ has the subspace topology of the topology considered in (b).

In effect, $X_1$ is opened up at $(0, 0)$ to make room to insert a copy of $\mathbb{R}$ there. $Y$ is made up of an open triangle with $(0, 0)$ as one vertex. The triangle is opened up at this vertex to make room to insert the copy of $(-1, 1)$ giving the rectangle $Z$.

18. We can extend the ideas of Exercise 17 in two ways: replace $X_1$ by all of $\mathbb{R}^2$ except the $y$-axis; open up the now doubled $X_1$ at each point of the $y$-axis and insert a copy of $\mathbb{R}$ at each of these points. The construction is very similar to that in Exercise 17. Let

$$N_{a,r} = \{(x, y) \in \mathbb{R}^2 \ / \ 0 < |x| < r \text{ and } (a-r)|x| < y < (a+r)|x|\} \cup \{0\} \times \{0\} \times (a-r, a+r).$$

With $X_1 = \{(x, y) \in \mathbb{R}^2 \ / \ y \neq 0\}$ and $X = X_1 \cup \{0\} \times \mathbb{R} \times \mathbb{R}$, let $T_\eta : X \to X$ be defined for each $\eta \in \mathbb{R}$ by $T_\eta(x, y) = (x, y + \eta)$ and $T_\eta(0, y, z) = (0, y + \eta, z)$. Let

$$\mathcal{B} = \{U \cap X_1 \ / \ U \text{ is open in the usual topology on } \mathbb{R}^2\} \cup \{T_\eta(N_{a,r}) \ / \ a, \eta \in \mathbb{R} \text{ and } r > 0\}.$$

(a) Draw pictures of $T_\eta(N_{a,r})$ for two different values of $(\eta, a, r)$.

(b) Prove that $\mathcal{B}$ is a basis for a topology on $X$.

(c) Prove that if $X$ is topologised using the basis $\mathcal{B}$ then each point of $X$ has an open neighbourhood homeomorphic to $\mathbb{R}^2$.