The Fourier Integral (17.9)

The Fourier series is useful for functions defined on a finite interval. For functions defined on the whole line $-\infty < x < \infty$, the Fourier series is no longer useful (unless the function is periodic). In this case we can represent the function as a Fourier integral.

$$\boxed{f(x) = \int_0^\infty a(\omega)\cos\omega x + b(\omega)\sin\omega x \, d\omega}$$

where
$$\boxed{\begin{aligned} a(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x \, dx \\ b(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x \, dx \end{aligned}}$$

Note the similarity with Fourier series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx,$$
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

The variable ω is called the *frequency*. The frequency varies in the interval $[0,\infty)$. Notice that there are frequencies associated with the Fourier series as well,

$$\omega_n = \frac{n\pi}{l}, \quad n = 0, 1, 2, 3 \ldots$$

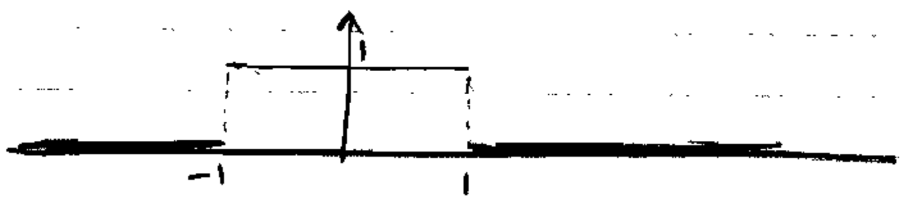
These frequencies are discrete, but they are separated by $\Delta\omega = \frac{\pi}{l}$ which gets smaller as $l \to \infty$.

Fourier Integral Theorem

Suppose that f is defined on $(-\infty, \infty)$ and
(i) f and f' are piecewise continuous on every finite interval
(ii) $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$

Then the Fourier integral of f converges to $f(x)$ if f is continuous at x and it converges to
$\frac{1}{2}\left(f(x^-) + f(x^+)\right)$
is f is not continuous at x.

Example (Rectangular Pulse)



$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = \frac{1}{\pi}\int_{-1}^{1}\cos\omega x\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega x}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\omega}$$

$b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = 0$ because f is even.

So the Fourier integral representation of f is

$$f(x) = \int_0^{\infty} a(\omega)\cos\omega x + b(\omega)\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\cos\omega x\,d\omega$$

Note that at $x = \pm 1$ this converges to $\frac{1}{2}(0+1) = \frac{1}{2}$.

<u>Outline</u> of <u>the</u> proof of <u>the Fourier integral theorem</u>

The idea is to write down the Fourier series for f on the interval $[-l, l]$ and see what happens as $l \to \infty$.

The Fourier series of f on $[-l, l]$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(y)\,dy + \sum_{n=1}^{\infty}\left[\frac{1}{l}\int_{-l}^{l} f(y)\cos\frac{n\pi y}{l}\,dy\,\cos\frac{n\pi x}{l}\right.$$

$$\left. + \frac{1}{l}\int_{-l}^{l} f(y)\sin\frac{n\pi y}{l}\,dy\,\sin\frac{n\pi x}{l}\right]$$

use $\cos A\cos B + \sin A \sin B = \cos(A-B)$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(y)\,dy + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(y)\cos\frac{n\pi}{l}(y-x)\,dy$$

$$= \frac{1}{2l}\int_{-l}^{l} f(y)\,dy + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-l}^{l} f(y)\cos\omega_n(y-x)\,dy\,\Delta\omega$$

where $\omega_n = \frac{n\pi}{l}$ and $\Delta\omega = \frac{\pi}{l}$.

The first term $\to 0$ as $l \to \infty$ because

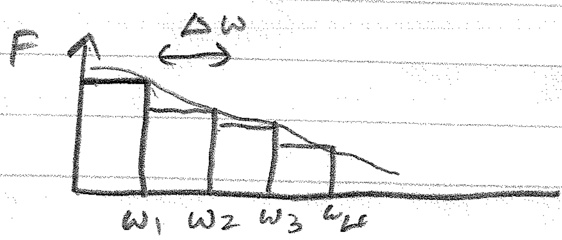
$$\left|\frac{1}{2l}\int_{-l}^{l} f(y)\,dy\right| \le \frac{1}{2l}\int_{-\infty}^{\infty}|f(y)|\,dy \to 0 \quad \text{as } l \to \infty.$$

Also $\int_{-l}^{l} f(y)\cos\omega_n(y-x)\,dy \to \int_{-\infty}^{\infty} f(y)\cos\omega_n(y-x)\,dy$ as $l \to \infty$.

So
$$f(x) = \sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\cos\omega_n(y-x)\,dy\,\Delta\omega$$

$$= \sum_{n=1}^{\infty} F(\omega_n)\,\Delta\omega \qquad F(\omega_n) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\cos\omega_n(y-x)\,dy$$

$$\to \int_0^{\infty} F(\omega)\,d\omega \quad \text{as } l \to \infty \text{ because } \Delta\omega = \frac{\pi}{l}$$



Hence

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos\omega(y-x)\,dy\,d\omega$$

Now rewriting $\cos\omega(y-x) = \cos\omega y\cos\omega x + \sin\omega y\sin\omega x$ gives

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(y)\cos\omega y\cos\omega x + f(y)\sin\omega y\sin\omega x\,dy\,d\omega$$

$$= \int_0^\infty a(\omega)\cos\omega x + b(\omega)\sin\omega x\,d\omega$$

The Fourier Transform (17.10)

This is the complex form of the Fourier integral but it is used more often than the Fourier integral.

The Fourier transform is derived from the Fourier integral in just the same way that the complex Fourier series is derived from the trigonometric fourier series. I won't include the details here. Take a look at Section 17.10 of the text if you are interested.

The Fourier transform of a function $f(x)$ is denoted $F\{f(x)\}$ or $\hat{f}(\omega)$ and is given by

$$\boxed{\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx}$$

Recall that $a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx$ and $b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx$. Since

$$e^{-i\omega x} = \cos\omega x - i\sin\omega x$$

we see that $\hat{f}(\omega) = \pi(a(\omega) - i b(\omega))$.

The inverse transform is denoted $F^{-1}\{\hat{f}(\omega)\}$ and it gives us $f(x)$

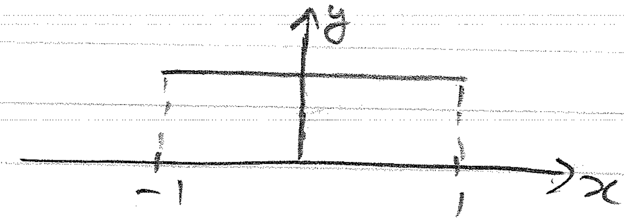
$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega}$$

This formula is the complex version of the Fourier integral

$$f(x) = \int_{0}^{\infty} a(\omega)\cos\omega x + b(\omega)\sin\omega x\, dx.$$

Examples

1. Rectangular pulse



$$\hat{f}(\omega) = \int_{\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-1}^{1} e^{-i\omega x}\, dx$$

$$= \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1} = \frac{-1}{i\omega}\left(e^{-i\omega} - e^{i\omega}\right)$$

$$= \frac{1}{i\omega}\left(e^{i\omega} - e^{-i\omega}\right)$$

$$= \frac{2\sin\omega}{\omega}$$

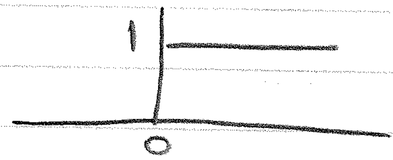
The inversion formula tells us that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin\omega}{\omega}\, e^{i\omega x}\, d\omega$$

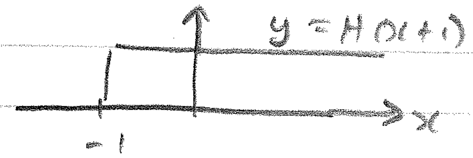
Note It's useful to define the Heaviside function

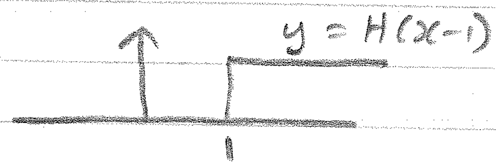
$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



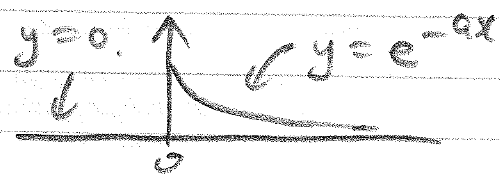
Notice that the rectangular pulse is given by

$$f(x) = H(x+1) - H(x-1)$$





2. $f(x) = H(x)\, e^{-ax}$



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} H(x) e^{-ax} e^{-i\omega x} dx$$

$$= \int_0^{\infty} e^{-ax} e^{-i\omega x} dx$$

$$= \int_0^{\infty} e^{-(a+i\omega)x} dx$$

$$= -\frac{1}{a+i\omega} \left[e^{-(a+i\omega)x} \right]_0^{\infty}$$

$$= \frac{1}{a+i\omega}$$

Properties of the transform

1. Linearity $\quad F\{\alpha f(x) + \beta g(x)\} = \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$
and similarly for the inverse transform.

2. Derivatives

$$F\{f'(x)\} = i\omega \hat{f}(\omega)$$

Proof $\quad F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx$

(Integrating by parts) $= \left[f(x) e^{-i\omega x} \right]_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$$= i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= i\omega \hat{f}(\omega)$$

Hence $\quad F\{f''(x)\} = i\omega F\{f'(x)\} = i\omega\, i\omega \hat{f}(\omega)$
$= (i\omega)^2 \hat{f}(\omega)$

Clearly $\quad F\{f^{(n)}(x)\} = (i\omega)^n \hat{f}(\omega).$

Example Heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty$
$u(x,0) = f(x)$. Let $\hat{u}(\omega, t) = \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx.$
Then $\frac{d\hat{u}}{dt} = -\omega^2 \hat{u}$ (An ODE!). More on this later.

3. Fourier Convolution

This is defined to by

$$f*g\,(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

Note that this is slightly different from the Laplace convolution $f*g\,(x) = \int_0^x f(x-y)\,g(y)\,dy$ but if f and g are both zero on $(-\infty, 0]$ then the Laplace convolution equals the Fourier convolution

The Fourier convolution theorem is proved in the same way that we proved the Laplace convolution theorem:

$$F\{f*g\,(x)\} = \hat{f}(\omega)\,\hat{g}(\omega)$$

4. Translation Formulas

We saw this with the Laplace transform and it also works for F and F^{-1}:

$$F\{f(x-a)\} = e^{-ia\omega}\hat{f}(\omega)$$

Similarly $F^{-1}\{\hat{f}(\omega-a)\} = e^{ia\omega} f(x)$

Check $F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)\,e^{-i\omega x}\,dx$

Let $y = x-a$ $\quad dy = dx$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(y)\,e^{-i\omega(y+a)}\,dy$$

$$= e^{-i\omega a}\int_{-\infty}^{\infty} f(y)\,e^{-i\omega y}\,dy$$

$$= e^{-i\omega a}\hat{f}(\omega)$$