A few more Laplace transforms

Example $\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt$

We could integrate by parts or just notice that

$$\int_0^\infty t e^{-st}\,dt = -\frac{d}{ds}\int_0^\infty e^{-st}\,dt$$

$$= -\frac{d}{ds}\left(\frac{1}{s}\right)$$

$$= \frac{1}{s^2}$$

Similarly $\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt$

$$= -\frac{d}{ds}\int_0^\infty t e^{-st}\,dt$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2}\right)$$

$$= \frac{2}{s^3}$$

If you keep doing this, you find

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

A useful rule If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at} f(t)\} = f(s-a)$

Proof $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

Example $\mathcal{L}\{t e^{at}\} = F(s-a)$ where $F(s) = \frac{1}{s^2}$

$$= \frac{1}{(s-a)^2}$$

In fact $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

Example $\mathcal{L}\{e^{t}\sin 2t\} = F(s-1)$

where $F(s) = \mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2+4}$

so $\mathcal{L}\{e^{t}\sin 2t\} = \dfrac{2}{(s-1)^2+4}$

Applications to the Solution of Differential Equations (5.4)

We make use of:

$$\mathcal{L}\{f(t)\} = F(s)$$
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

etc.

Example

$$y^{(iv)} - y = 0, \quad y(0)=1, \; y'(0) = y''(0) = y'''(0) = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$$s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y = 0$$
$$(s^4-1)Y = s^3$$
$$Y(s) = \frac{s^3}{s^4-1} = \frac{s^3}{(s^2-1)(s^2+1)} = \frac{s^3}{(s-1)(s+1)(s^2+1)}$$

So write

$$\frac{s^3}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$s^3 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+1)$$

Let $s=1$ $\quad 1 = A.4$ $\quad$ so $A = \frac{1}{4}$

$s=-1$ $\quad -1 = B.(-4)$ $\quad$ so $B = \frac{1}{4}$

coeff s^3 $\quad 1 = A + B + C$ $\quad C = 1/2$

coeff 1 $\quad 0 = A - B - D$ $\quad D = 0$

$$Y(s) = \frac{1/4}{s-1} + \frac{1/4}{s+1} + \frac{\frac{1}{2}s}{s^2+1}$$

$$y(t) = \tfrac{1}{4}e^{t} + \tfrac{1}{4}e^{-t} + \tfrac{1}{2}\cos t$$
$$= \tfrac{1}{2}\cosh t + \tfrac{1}{2}\cos t$$

<u>Note</u> $s^2+1 = (s+i)(s-i)$ so we could have used

$$\frac{s^3}{(s+1)(s-1)(s+i)(s-i)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+i} + \frac{D}{s-i}$$

<u>Example</u>

$$x'' - 4x' - 5x = e^{-t} \qquad x(0) = x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$(s^2 - 4s - 5)X(s) = \frac{1}{s+1}$$

$$(s-5)(s+1)\,X(s) = \frac{1}{s+1}$$
$$X(s) = \frac{1}{(s-5)(s+1)^2} = \frac{A}{s-5} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multiply by $(s-5)(s+1)^2$

$$1 = A(s+1)^2 + B(s-5)(s+1) + C(s-5)$$

Let $s=-1$ $\quad 1 = -6C \quad$ so $C = -\frac{1}{6}$
Let $s=5$ $\quad 1 = 36A \quad$ so $A = \frac{1}{36}$

coefficient of s^2: $\quad 0 = A + B \quad$ so $B = -\frac{1}{36}$

$$X(s) = \frac{1}{36}\frac{1}{s-5} - \frac{1}{36}\frac{1}{s+1} - \frac{1}{6}\frac{1}{(s+1)^2}$$

$$x(t) = \frac{1}{36}e^{5t} - \frac{1}{36}e^{-t} - \frac{1}{6}te^{-t}$$