Laplace Transforms (5.1, 5.2)

Definition The Laplace transform of a function $f(t)$ defined for $t \geq 0$ is

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

Some examples:

$f(t) = 1$ $\quad F(s) = \int_0^\infty e^{-st}\, dt$

$$= \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty}$$

$$= \frac{1}{s} \quad \text{(we need } s > 0\text{)}$$

(also works if $\operatorname{Re} s > 0$)

$f(t) = e^{at}$ $\quad F(s) = \int_0^\infty e^{-st} e^{at}\, dt$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=0}^{t=\infty}$$

$$= \frac{1}{s-a} \quad \text{if } s > a$$

Other notation $\mathcal{L}\{f(t)\}$ also denotes the Laplace transform

Note that $\mathcal{L}$ is linear $\mathcal{L}\{\alpha f(t) + \beta g(t)\}$
$= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$

Example $\mathcal{L}\{3 + 4e^{2t}\} = 3\mathcal{L}\{1\} + 4\mathcal{L}\{e^{2t}\}$

$$= \frac{3}{s} + \frac{4}{s-2}$$

What can we do with Laplace tranforms?

We'll see that the Laplace transform of a derivative $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ where $F(s) = \mathcal{L}\{f(t)\}$. This allows us to convert differential equations to algebraic equations. We then solve the algebraic equation and thus find a solution to the differential equation. More on this soon!

More Laplace transforms:

$$f(t) = \cos at$$

$$F(s) = \int_0^\infty \cos at \; e^{-st}\, dt$$

We could integrate by parts, but it's easier to use

$$e^{iat} = \cos at + i \sin at$$

$$\text{so } \cos at = \operatorname{Re} e^{iat}$$

$$\text{so } \mathcal{L}(\cos at) = \operatorname{Re} \mathcal{L}(e^{iat})$$

$$\mathcal{L}(e^{iat}) = \int_0^\infty e^{iat} e^{-st}\, dt$$

$$= \int_0^\infty e^{(ia-s)t}\, dt$$

$$= \left[\frac{e^{(ia-s)t}}{ia-s} \right]_0^\infty$$

$$= \frac{1}{s-ia}$$

$$= \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$\mathrm{Re}\ \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2}$$

So $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

Notice $\sin at = \mathrm{Im}\ e^{iat}$

So $\mathcal{L}\{\sin at\} = \mathrm{Im}\ \dfrac{s+ia}{s^2+a^2}$

$$= \frac{a}{s^2+a^2}.$$

__Definition__ $f(t)$ is of exponential order as $t \to \infty$ if there exist real constants K, c and T such that

$$|f(t)| \leq K e^{ct} \quad \text{for all } t \geq T.$$

__Note__ If $f(t)$ is continuous for $t \geq 0$ and of exponential order then it's Laplace transform exists for $s > c$ because

$$\int_0^\infty f(t) e^{-st}\, dt = \underbrace{\int_0^T f(t) e^{-st}\, dt}_{\text{defined}}$$

$$+ \int_T^\infty f(t) e^{-st}\, dt$$

and $\displaystyle\int_T^\infty |f(t)| e^{-st}\, dt \leq \int_T^\infty K e^{(c-s)t}\, dt$

which is convergent for $s > c$.

__Note__ We don't need f to be continuous. e.g. piecewise continuous will do.

Properties of the transform (5.3)

We've already noted that the transform is linear, and this helps us calculate transforms. But it also helps us calculate the original function from its transform.

Example $F(s) = \dfrac{s+1}{s^2+2s-3}$

Find $f(t)$.

Use partial fractions.

$$\frac{s+1}{s^2+2s-3} = \frac{s+1}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

$$\text{so } s+1 = A(s-1) + B(s+3)$$

Put $s=1$. Then $B = \frac{2}{4} = \frac{1}{2}$

$s=-3$. Then $A = \frac{-2}{-4} = \frac{1}{2}$

Hence $F(s) = \dfrac{\frac{1}{2}}{s+3} + \dfrac{\frac{1}{2}}{s-1}$

so $f(t) = \frac{1}{2}e^{-3t} + \frac{1}{2}e^{t}$

Transforms of derivatives

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Proof $\int_0^\infty f'(t)e^{-st}\,dt = ?$

Integrate by parts. $du = f'(t)dt$

$v = e^{-st}$

$$u = f(t)$$
$$dv = -se^{-st}\,dt$$

so $$\int_0^\infty f'(t)e^{-st}\,dt = [uv]_0^\infty - \int_0^\infty u\,dv$$

$$= [f(t)e^{-st}]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}.$$

What about $\mathcal{L}\{f''(t)\}$. Since $f''(t)$ is the derivative of $f'(t)$, we can apply the result for first derivatives twice:

$$\begin{aligned} \mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s[\,s\mathcal{L}\{f(t)\} - f(0)\,] - f'(0) \\ &= s^2\mathcal{L}\{f(t)\} - f'(0) - s\,f(0) \end{aligned}$$

Similarly for higher order derivatives.

Let's summarise: If $\mathcal{L}\{f(t)\} = F(s)$

then
$$\begin{aligned} \mathcal{L}\{f(t)\} &= F(s) \\ \mathcal{L}\{f'(t)\} &= sF(s) - f(0) \\ \mathcal{L}\{f''(t)\} &= s^2F(s) - sf(0) - f'(0) \\ \mathcal{L}\{f'''(t)\} &= s^3F(s) - s^2f(0) - sf'(0) - f''(0) \end{aligned}$$

etc.

This is useful for solving differential equations

Example Solve

$$x'' + x = 1 \qquad x(0) = 1, \quad x'(0) = 0$$

Take the Laplace transform of each side.

Let $F(s) = \mathcal{L}\{x(t)\}$

$$\mathcal{L}\{x''(t)\} + \mathcal{L}\{x(t)\} = \mathcal{L}\{1\}$$

$$s^2F(s) - s\,x(0) - x'(0) + F(s) = \frac{1}{s}$$

$$(s^2+1)F(s) - s = \frac{1}{s}$$

$$(s^2+1)F(s) = \frac{1}{s} + s = \frac{s^2+1}{s}$$

$$F(s) = \frac{1}{s}$$

So $x(t) = 1$

Laplace convolution theorem

Definition The Laplace convolution of two functions $f(t)$ and $g(t)$ defined for $t \geq 0$ is given by

$$f * g\,(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Example $f(t) = \sqrt{t}$, $g(t) = t$

$$f*g\,(t) = \int_0^t \sqrt{\tau}\,(t-\tau)\,d\tau$$

$$= \int_0^t t\tau^{1/2} - \tau^{3/2}\,d\tau$$

$$= \left[\frac{2}{3}t\,\tau^{3/2} - \frac{2}{5}\tau^{5/2}\right]_{\tau=0}^{\tau=t}$$

$$= \left(\frac{2}{3} - \frac{2}{5}\right)t^{5/2}$$

$$= \frac{4}{15}t^{5/2}$$

Laplace Convolution Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$
then

$$\mathcal{L}\{f * g(t)\} = F(s)\,G(s)$$

Example $\dfrac{1}{s(s^2+1)} = F(s)\,G(s)$

where $F(s) = \dfrac{1}{s^2+1}$ $G(s) = \dfrac{1}{s}$

These have inverse transforms

$$f(t) = \sin t \qquad g(t) = 1$$

So $\dfrac{1}{s(s^2+1)}$ has inverse transform

$$\begin{aligned} f*g(t) &= \int_0^t f(\tau)\,g(t-\tau)\,d\tau \\ &= \int_0^t \sin\tau\,d\tau \\ &= \left[-\cos\tau\right]_0^t \\ &= 1-\cos t \end{aligned}$$

Example $\dfrac{1}{(s+1)(s+2)}$ $F(s) = \dfrac{1}{s+1}$ $G(s) = \dfrac{1}{s+2}$

$f(t) = e^{-t}$ $g(t) = e^{-2t}$

$$\begin{aligned} f*g(t) &= \int_0^t f(\tau)\,g(t-\tau)\,d\tau \\ &= \int_0^t e^{-\tau}\,e^{-2(t-\tau)}\,d\tau \\ &= e^{-2t}\int_0^t e^{\tau}\,d\tau = e^{-2t}\left[e^{\tau}\right]_{\tau=0}^{\tau=t} \\ &= e^{-t} - e^{-2t} \end{aligned}$$

So the inverse transform of $\frac{1}{(s+1)(s+2)}$ is

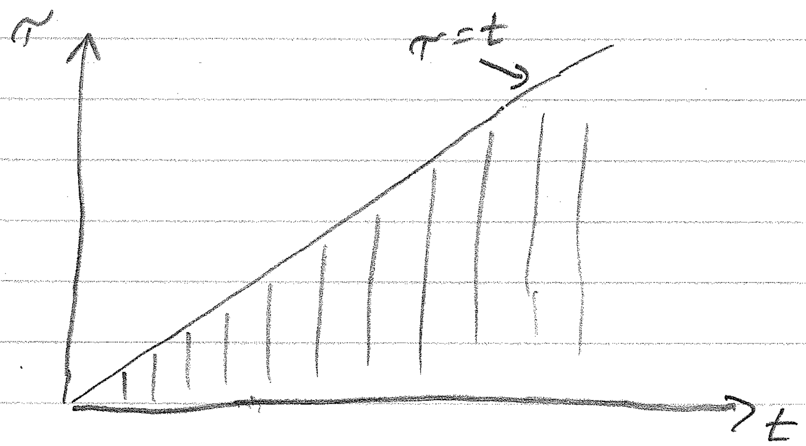
$e^{-t} - e^{-2t}$

i.e. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = e^{-t} - e^{-2t}$

<u>Exercise</u> Derive this using partial fractions.

<u>Proof of the convolution theorem</u>

$$\mathcal{L}\{f*g\,(t)\} = \int_0^\infty \left(\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right) e^{-st}\, dt$$



Interchange order of integration.

$$= \int_0^\infty \int_\tau^\infty f(\tau)\, g(t-\tau)\, e^{-st}\, dt\, d\tau$$

$$= \int_0^\infty f(\tau) \int_\tau^\infty g(t-\tau) e^{-st}\, dt\, d\tau$$

The inside integral $= \int_\tau^\infty g(t-\tau)\, e^{-st}\, dt$

Set $\mu = t - \tau \qquad d\mu = dt$

So $\int_\tau^\infty g(t-\tau)\, e^{-st}\, dt = \int_0^\infty g(\mu)\, e^{-s(\mu+\tau)}\, d\mu$

$$= e^{-s\tau} \int_0^\infty g(\mu)\, e^{-s\mu}\, d\mu$$

$$= e^{-s\tau}\, G(s)$$

So $\mathcal{L}\{f*g\,(t)\} = \int_0^\infty f(\tau)\, e^{-s\tau}\, G(s)\, d\tau$

$= F(s)\, G(s).$

<u>Note</u> The theorem implies that $f*g = g*f$ (commutative)