Sturm-Liouville Theory, continued.

What about the sign of the eigenvalues? Let's do what we did before for $y'' + \lambda y = 0$

$$(p(x)y')' + q(x)y + \lambda w(x) y = 0$$

multiply by y and integrate

$$\int_a^b y\,(p(x)y')'\,dx + \int_a^b (q(x) + \lambda w(x))\,y^2\,dx = 0$$

Integrate first part by parts

$$\int_a^b y\,(p(x)y')'\,dx = \left[y\,(p(x)y')\right]_a^b - \int_a^b p(x)(y')^2\,dx$$

So $$\lambda \int_a^b w(x)\,y^2\,dx = -\left[p(x)\,y\,y'\right]_a^b + \int_a^b p(x)(y')^2 - q\,y^2\,dx$$

so $$\lambda = \frac{-\left[p(x)yy'\right]_a^b + \int_a^b p(x)(y')^2 - q(x)\,y^2\,dx}{\int_a^b w(x)\,y^2\,dx}$$

This is called the <u>Rayleigh</u> <u>quotient</u>. (It isn't discussed in the text)

Denominator > 0 so $\lambda \geq 0$ if numerator ≥ 0.
Notice that this will be the case if
$p(x) > 0$, $q(x) \leq 0$ and $\left[p(x)yy'\right]_a^b \leq 0$.

In many problems we can check these details.

<u>Example</u> (Example 3, Page 893 in text)

$$y^4 + \lambda y = 0$$
$$y(0) - 2y'(0) = 0, \quad y(1) = 0.$$

In this case, $w(x) = 1$, $q(x) = 0$, $p(x) = 1$

$$[p(x)y'y]_0^1 = [y'y]_0^1 = y'(1)y(1) - y'(0)y(0)$$
$$= -y'(0)y(0)$$
$$= -\tfrac{1}{2}y(0)y(0)$$
$$= -\tfrac{1}{2}y(0)^2 \leq 0$$

So the eigenvalues are ≥ 0.

So we can write $\lambda = k^2$

Is there an zero eigenvalue?

Need $y'' = 0$ $\quad y = c_1 + c_2 x$

But $y(1) = 0 \Rightarrow c_1 + c_2 = 0$ So $c_2 = -c_1$

$y(x) = c_1 - c_1 x$

$y'(x) = -c_1$

$y(0) - 2y'(0) = 0 \Rightarrow c_1 + 2c_1 = 0$

So $c_1 = 0$, $y = 0$ so no zero e-value!

So $\lambda = k^2$, $k > 0$

$y = c_1 \cos kx + c_2 \sin kx$

$y' = -c_1 k \sin kx + c_2 k \cos kx$

$y(1) = 0 \Rightarrow c_1 \cos k + c_2 \sin k = 0$

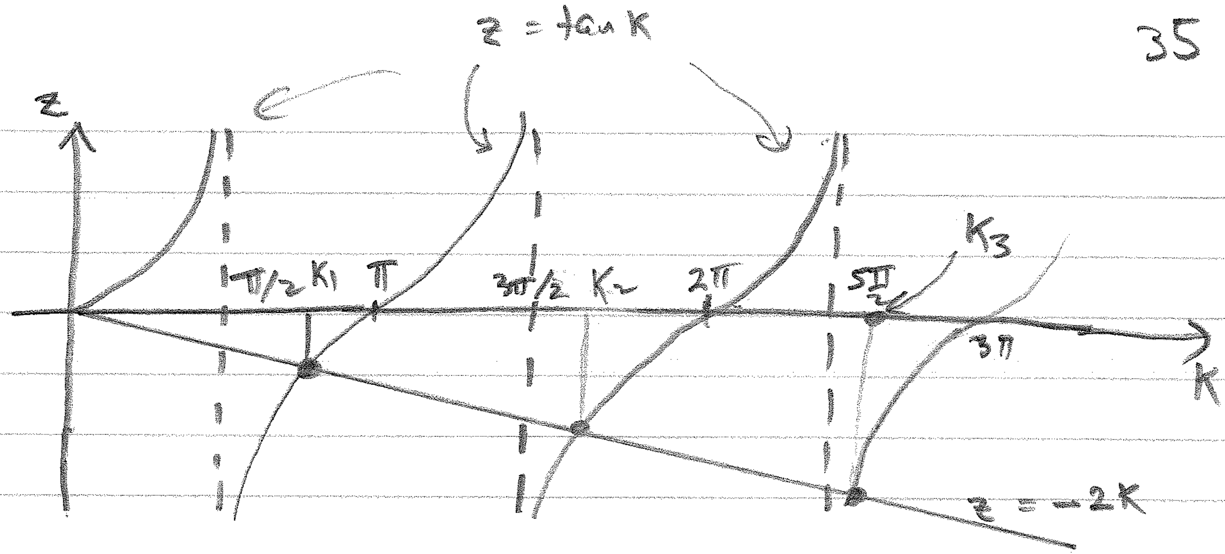
$y(0) - 2y'(0) = 0 \quad c_1 - 2c_2 k = 0$

$$\begin{bmatrix} \cos k & \sin k \\ 1 & -2k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For non-trivial solutions the determinant of the coefficient matrix must be zero

i.e. $-2k \cos k - \sin k = 0$

$$\tan k = -2k$$



Notice that the roots $K_1, K_2, K_3, \ldots$ satisfy $\frac{\pi}{2} < K_1 < \pi$, $\frac{3\pi}{2} < K_2 < 2\pi$, $\frac{5\pi}{2} < K_2 < 3\pi$ etc.

But $\lambda = K^2$ so

$$\left(\frac{\pi}{2}\right)^2 < \lambda_1 < \pi^2, \quad \left(\frac{3\pi}{2}\right)^2 < \lambda_2 < 4\pi^2, \quad \text{etc.}$$

To find good approximations to the eigenvalues we need to use numerical methods like Newton's method.

Using Sturm-Liouville Theory for PDEs.
(Example 4, 18.3.3)

Solve $u_t = u_{xx} \quad 0<x<1, \quad 0<t<\infty$

$u(0,t) - 2u_x(0,t) = 5$

$u(1,t) = 35$

$u(x,0) = f(x)$

Steady-state sol.

$u_{xx} = 0$ so $u = c_1 + c_2 x$

$u(0) - 2u'(0) = 5 \Rightarrow c_1 - 2c_2 = 5$

$u(1) = 35 \Rightarrow c_1 + c_2 = 35$

So $3c_2 = 30 \Rightarrow c_2 = 10, \quad c_1 = 25$

i.e. $u_s(x) = 25 + 10x$.

Now let $\bar{u}(x,t) = u(x,t) - u_s(x)$

$$\bar{u}(x,0) = u(x,0) - u_s(x) = f(x) - 25 - 10x = g(x)$$

$\bar{u}(0,t) - 2\bar{u}_x(0,t) = 0$

$\bar{u}(1,t) = 0$.

Special solutions: $\bar{u}(x,t) = y(x)\,T(t)$

$$y(x)\,T'(t) = y''(x)\,T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{y''(x)}{y(x)} = -\lambda$$

So $y'' + \lambda y = 0$

B.C. give $y(0) - 2y'(0) = 0$

$y(1) = 0$

But we've already used Sturm theory on this (see last lecture)

$y = c_1 \cos Kx + c_2 \sin Kx$

where $\tan K = -2K$, $\lambda = K^2$

There are positive e-values $\lambda_1, \lambda_2, \ldots$
$\lambda_n = K_n^2$

Note that $y(0) - 2y'(0) = 0$
gives $c_1 - 2Kc_2 = 0$
so $c_1 = 2Kc_2$

Hence $y = 2Kc_2 \cos Kx + c_2 \sin Kx$
$= c_2(2K \cos Kx + \sin Kx)$

so the n^{th} eigenfunction is

$$\Phi_n(x) = 2K_n \cos K_n x + \sin K_n x.$$

$T(t) = ce^{-\lambda t} = ce^{-K_n^2 t}$ so we can write

$$U(x,t) = \sum_{n=1}^{\infty} c_n \Phi_n(x) e^{-K_n^2 t}$$

Need $U(x,0) = g(x)$ so

$$\sum_{n=1}^{\infty} c_n \Phi_n(x) = g(x)$$

Sturm theory says that the ϕ_n's are orthogonal and that we can expand $g(x)$ in this series.

In fact, multiplying each side by $\phi_M(x)$ and integrating gives

$$c_M \int_0^1 \phi_n(x)\phi_M(x)\,dx = \int_0^1 g(x)\,\phi_M(x)\,dx$$

so

$$c_M = \frac{\int_0^1 (f(x) - 25 - 10x)(2k_n\cos k_n x + \sin k_n x)\,dx}{\int_0^1 (2k_n\cos k_n x + \sin k_n x)^2\,dx}$$

(we could evaluate this if we knew f)

and $u(x,t) = U(x,t) + u_s(x)$

$$= \sum_{n=1}^{\infty} c_n (2k_n\cos k_n x + \sin k_n x)e^{-k_n^2 t}$$

$$+ 25 + 10x.$$