Complex Fourier Series (17.3.4)

Recall $e^{i\theta} = \cos\theta + i\sin\theta$

so $e^{-i\theta} = \cos\theta - i\sin\theta$

add $\rightarrow$ $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

substract $\rightarrow$ $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

We can rewrite the sin and cos terms in our Fourier series in terms of these complex exponentials.

Consider the sum in a Fourier series up to $n=N$:

$$a_0 + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$= a_0 + \sum_{n=1}^{N} \frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)$$

$$= a_0 + \sum_{n=1}^{N} \tfrac{1}{2}(a_n - ib_n)e^{\frac{in\pi x}{L}} + \tfrac{1}{2}(a_n + ib_n)e^{-\frac{in\pi x}{L}}$$

$$= a_0 + \sum_{n=1}^{N} \tfrac{1}{2}(a_n - ib_n)e^{\frac{in\pi x}{L}} + \sum_{n=-N}^{-1} a_{-n} + ib_{-n}\, e^{\frac{in\pi x}{L}}$$

$$= \sum_{n=-N}^{N} c_n e^{in\pi x/L}$$

where $c_0 = a_0$

$c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1$

$c_n = \frac{1}{2}(a_{-n} + ib_{-n}), \quad n \leq 1$

Notice, $c_0 = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x)\,dx$

If $n \geq 1$, $c_n = \frac{1}{2}(a_n - ib_n)$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{2l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{n\pi i x}{l}}\,dx$$

If $n \leq 1$,

$$c_n = \tfrac{1}{2}(a_{-n} + i b_{-n})$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx + \frac{i}{2l}\int_{-l}^{l} f(x)\sin\left(-\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx$$

In all cases

$$a_0 + \sum_{n=1}^{N} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l} = \sum_{n=-N}^{N} c_n e^{in\pi x/l}$$

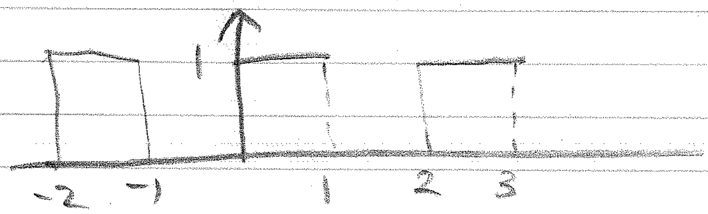
where $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx$

So we define the complex Fourier series of f to be

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where $\sum_{n=-\infty}^{\infty}$ is interpreted as $\lim_{N\to\infty}\sum_{n=-N}^{N}$

Example



Here $l = 2$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx$$

$$= \frac{1}{4}\int_0^1 e^{-in\pi x}\,dx$$

$$= \frac{1}{4}\left[\frac{-1}{in\pi}e^{-in\pi x}\right]_0^1 \quad \text{if } n \neq 0$$

$$= \frac{-1}{4i n\pi}\left(e^{-in\pi} - 1\right)$$

But $e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n$

So $c_n = \frac{1}{4in\pi}\left(1-(-1)^n\right) \qquad n \neq 0$

when $n=0$, $c_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$

$$= \frac{1}{2}\int_0^1 1\,dx$$

$$= \frac{1}{2}$$

So $f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{4in\pi} e^{in\pi x}$

<u>Sturm - Liouville Theory</u> (17.7)

Recall that when we separated variables we chose a separation constant $-k^2$, i.e. a non-positive constant. Let's see if we can prove that the constant is ≤ 0.

We had $X'' = -k^2 X$ or $X'' + k^2 X = 0$

But in general we might have

$$X''(x) + \lambda X(x) = 0 \quad \text{—— ①}$$

and we want to show $\lambda \geq 0$

We assume that X satisfies homogeneous boundary conditions such as

$$X(0) = 0, \quad X(L) = 0$$

Such a function X is called an <u>eigenfunction</u> with <u>eigenvalue λ</u>. (provided X is not identically zero).

It's not even clear from ① that λ is real so let's assume it is complex. Let $\bar{X}$ denote the complex conjugate of X

(Reminder: if $z = a+ib$ then $\bar{z} = a - ib$

$$z\bar{z} = a^2 + b^2 = |z|^2 \text{).}$$

Multiply ① by $\bar{X}$ and integrate

$$\lambda \int_0^L X \bar{X}\, dx = -\int_0^L \bar{X} X''\, dx$$

But

$$\begin{aligned} \bar{X} X'' &= (\bar{X} X')' - \bar{X}' X' \\ &= (\bar{X} X')' - |X'|^2 \end{aligned}$$

so

$$\begin{aligned} RHS &= -\int_0^L (\bar{X} X')' - |X'|^2\, dx \\ &= -[\bar{X} X']_0^L + \int_0^L |X'|^2 dx \\ &= \underline{-\bar{X}(L) X'(L) + \bar{X}(0) X'(0)} - \int_0^L |X'|^2 dx \end{aligned}$$

But if $X(L) = 0$ then $\bar{X}(L) = 0$
if $X(0) = 0$ then $\bar{X}(0) = 0$ } Dirichlet Boundary conditions.

so the underlined part is zero

Notice that the underlined part is zero also with Neumann boundary conditions $X'(L) = 0$, $X'(0) = 0$

We could also have a Dirichlet condition at one end and a Neumann condition at the other end.

In all these cases, we have

$$\lambda \int_0^L |X|^2 dx = \int_0^L |X'|^2 dx$$

so

$$\lambda = \frac{\int_0^L |X'|^2 dx}{\int_0^L |X|^2 dx}$$

This is ≥ 0 because numerator ≥ 0 and denominator > 0.

Is it possible to have $\lambda = 0$? For that we would need $\int_0^L |X'|^2 dx = 0$ and that could only happen if $X' = 0$ everywhere, so $X(x) \equiv c$, a constant.

If we had also at least one homogeneous Dirichlet condition, $X(0) = 0$ or $X(L) = 0$, then we would conclude $c = 0$ so $X(x) = 0$, which is not allowed.

Hence:

If $X(0) = 0$ or $X(L) = 0$, λ must be positive.

But if $X'(0) = 0$ and $X'(L) = 0$ are the boundary conditions then we obtain an eigenfunction

$$X(x) = 1 \quad \text{(or any other non-zero constant)}$$

and $\lambda = 0$.

Example Find the eigenfunctions and eigenvalues of

$$X'' + \lambda X = 0, \qquad X'(0) = X'(L) = 0$$

Solution We already know that $\lambda = 0$ is an eigenvalue with eigenfunction

$$X(x) = 1$$

Other eigenvalues are positive, so write $\lambda = K^2$

$$X'' + K^2 X = 0 \qquad K > 0$$

$$X = c_1 \cos Kx + c_2 \sin Kx$$
$$X' = -c_1 K \sin Kx + c_2 K \cos Kx$$

$X'(0) = 0 \Rightarrow c_2 = 0$ so $X = c_1 \cos Kx$

$X'(L) = 0 \Rightarrow -c_1 K \sin KL = 0$

But $c_1 \neq 0$ because X is not the trivial function

So $\sin KL = 0$

so $KL = n\pi$

$K = \frac{n\pi}{L}$

and the eigenfunctions are

$$X_n(x) = \cos \frac{n\pi}{L}$$

with $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 0, 1, 2, 3, \ldots$

We need only take $n \geq 0$ because $n=0$ gives the zero-eigenvalue and eigenfunction, n negative duplicates what we get for $n>0$.

Orthogonality

We showed orthogonality directly before

e.g. $\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} \frac{L}{2}, & m=n \\ 0, & m \neq n \end{cases}$

But it's easy to see why eigenfunctions are always orthogonal.

Let ϕ_m and ϕ_n be eigenfunctions of

$$X'' + \lambda X = 0$$

with eigenvalues λ_m and λ_n, with $\lambda_m \neq \lambda_n$

i.e. $\phi_n'' + \lambda_n \phi_n = 0$ ——— (2)

$\phi_m'' + \lambda_m \phi_m = 0$ ——— (3)

$\phi_m \times$ (2) $-$ $\phi_n \times$ (3) gives

$$\phi_m \phi_n'' - \phi_n \phi_m'' + (\lambda_n - \lambda_m)\phi_n \phi_m$$

Hence $\int_0^L \phi_n''\phi_m - \phi_m''\phi_n \, dx = (\lambda_m - \lambda_n)\int_0^L \phi_n\phi_m \, dx$ — (4)

Notice that

$$\frac{d}{dx}\left(\phi_n'\phi_m - \phi_m'\phi_n\right) = \phi_n''\phi_m + \cancel{\phi_n'\phi_m'} - \phi_m''\phi_n - \cancel{\phi_m'\phi_n'}$$

$$= \phi_n''\phi_m - \phi_m''\phi_n$$

So the LHS of (4) above is

$$\int_0^L \frac{d}{dx}\left(\phi_n'\phi_m - \phi_m'\phi_n\right) dx = \left[\phi_n'\phi_m - \phi_n\phi_m'\right]_0^L$$

$$= 0$$

If we have homogeneous Dirichlet or Neumann boundary conditions at $x = 0$ or L.

Note This also works for a homogeneous Robin boundary condition

$$X'(L) = -h_2 X(L)$$

or $$X'(0) = h_1 X(0),$$

as obtained for Newton-cooling at the end of a rod. In fact, Dirichlet, Neumann, Robin boundary conditions are all described by the equations

$$\alpha X(0) + \beta X'(0) = 0$$
$$\gamma X(L) + \delta X'(L) = 0$$

where α and β are not both zero; and γ and δ are not both zero.

The same mathematical techniques may be applied to more general second order d.e.'s described below:

General Sturm-Liouville Problem:

$$(p(x)y')' + q(x)y + \lambda w(x)y(x) = 0 \qquad a<x<b$$

with homogeneous boundary conditions:

$$\alpha y(a) + \beta y'(a) = 0$$

$$\gamma y(b) + \delta y'(b) = 0$$

Here $p(x) > 0$ and $w(x) > 0$ on $[a,b]$,
p, q, w continuous on $[a,b]$,
α, β not both zero
γ, δ " " ".

Sturm-Liouville Theorem

(a) The eigenvalues are real
(b) The eigenvalues are simple. i.e. each eigenvalue has only one linearly independent eigenfunction.
There is an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3, \ldots$$

where $\lambda_n \to \infty$ as $n \to \infty$

(c) Eigenfunctions corresponding to distinct eigenvalues are orthogonal, i.e.

$$\int_a^b \phi_m(x)\phi_n(x)w(x)\,dx = 0 \quad \text{if } m \neq n$$

(d) If f and f' are piecewise continuous on $[a,b]$ then we can form the eigenfunction expansion ("Fourier series")

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

where

$$a_n = \frac{\int_a^b \phi_n(x) f(x) w(x)\,dx}{\int_a^b (\phi_n(x))^2 w(x)\,dx}$$

which converges to $f(x)$ wherever f is continuous and to

$$\frac{1}{2}\left(f(x^-) + f(x^+)\right)$$

wherever f has a jump.