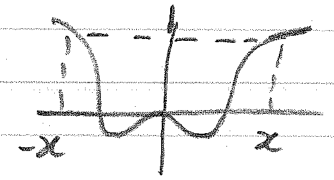
Fourier Series

These are series of sines and cosines such as those we found when solving the heat equation. We need to study these series more carefully.

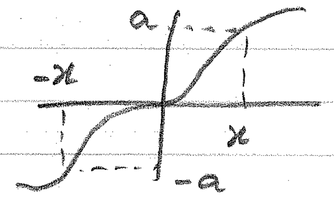
Preliminaries

Even, odd and periodic functions (17.2)

f is even if $f(x) = f(-x)$



f is odd if $f(-x) = -f(x)$



Notice that
even + even = even
odd + odd = odd
even × even = even
odd × odd = even
odd × even = odd

If f is odd then $\int_{-A}^{A} f(x)\,dx = 0$

If f is even then $\int_{-A}^{A} f(x)\,dx = 2\int_0^A f(x)\,dx$.

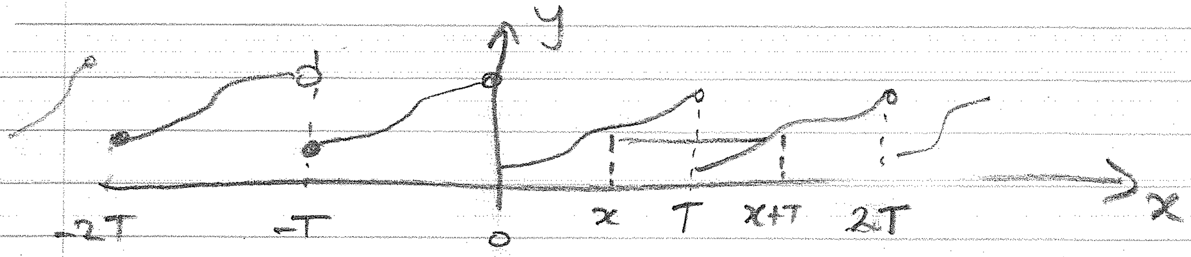
Any function can be written as a sum of even and odd functions:

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

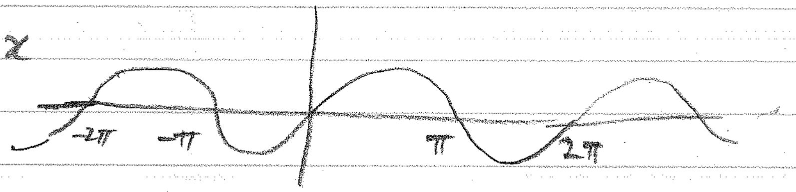
e.g. $f(x) = e^x = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2}$

$= \cosh(x) + \sinh(x)$

If $f(x+T) = f(x)$ we say that f is periodic with period T.



e.g. $\sin x$

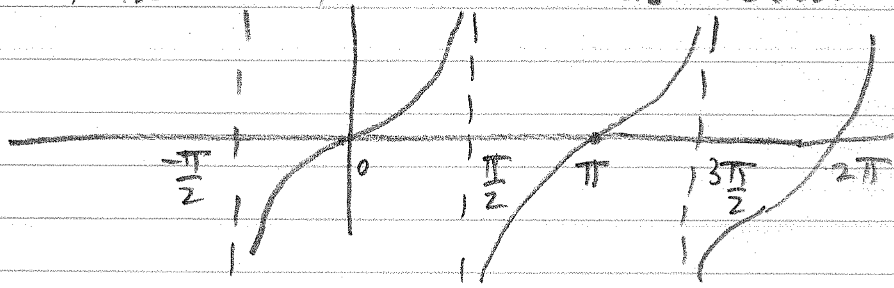


$\sin x$ is periodic with period 2π.

$\cos x$ " " " " 2π

$\tan x$ " " " " 2π

in fact $\tan x$ has period π as well.



If f has period T then it also has period $2T$, $3T$, etc. The smallest period of a function is called its fundamental period. $\tan x$ has fundamental period π.

<u>Fourier series of a periodic function.</u> 17.3.

The functions $\cos \frac{n\pi x}{l}$, $n = 0, 1, 2, \ldots$

and $\sin \frac{n\pi x}{l}$, $n = 1, 2, \ldots$ each have period $2l$

So if $f(x)$ has period $2l$, can we write

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$.

orthogonality equations

① $\int_{-l}^{l} \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l}\,dx = \begin{cases} 0 & m \neq n \\ l & m = n \end{cases}$

② $\int_{-l}^{l} \cos\frac{m\pi x}{l} \cos\frac{n\pi x}{l}\,dx = \begin{cases} 0 & m \neq n \\ l & m = n \neq 0 \\ 2l & m = n = 0. \end{cases}$

③ $\int_{-l}^{l} \sin\frac{m\pi x}{l} \cos\frac{n\pi x}{l}\,dx = 0.$

You can prove these the same way that we proved the equations for sine over the interval $[0, L]$ earlier. Note that ③ is obvious because the integrand is odd (odd = odd × even).

Hence if

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$

Formally, multiply each side by $\sin\frac{m\pi x}{l}$ and integrate from $-l$ to l. Because of the orthogonality, only the $n=m$ sine term contributes to the sum on the RHS so

$$\int_{-l}^{l} f(x) \sin \frac{m\pi x}{l}\, dx = \int_{-l}^{l} b_m \sin^2 \frac{m\pi x}{l}\, dx$$

$$= l\, b_m$$

so $$\boxed{b_m = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{m\pi x}{l}\, dx}$$

Similarly $$\boxed{a_m = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{m\pi x}{l}\, dx \qquad m \geq 1}$$

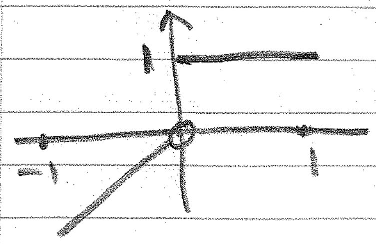
But a_0 is different. Integrate each side from $-l$ to l

$$\int_{-l}^{l} f(x)\, dx = \int_{-l}^{l} a_0\, dx = 2l a_0$$

so $$\boxed{a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\, dx = \text{average value of } f}$$

<u>Piecewise continuity</u> A function $f(x)$ is piecewise continuous on an interval $a \leq x \leq b$ if there exists a finite number of points $x_1, x_2, \ldots, x_N$ such that $f(x)$ is continuous on each open interval (a, x_1), (x_1, x_2), $(x_2, x_3), \ldots, (x_{N-1}, x_N)$, (x_N, b) and has a finite limit as x approaches each endpoint of these intervals from the insides of the intervals.

<u>Example</u> $f(x) = \begin{cases} x, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1 \end{cases}$



$f(x)$ is piecewise continuous because it is continuous on $(-1, 0)$ and on $(0, 1)$ and because as x

over

approaches -1 in $(-1,0)$, $f(x)$ approaches -1;
as x approaches 0 in $(-1,0)$, $f(x)$ " 0;
as x " 0 " $(0,1)$, " " 1;
as x " 1 " $(0,1)$, " " 1.

Notation The left limit of $f(x)$ at $x=c$ is denoted $f(c^-)$ and the right limit is denoted $f(c^+)$. Thus, in the example above,

$$f(-1^+) = -1$$
$$f(0^-) = 0$$
$$f(0^+) = 1$$
$$f(1^-) = 1$$

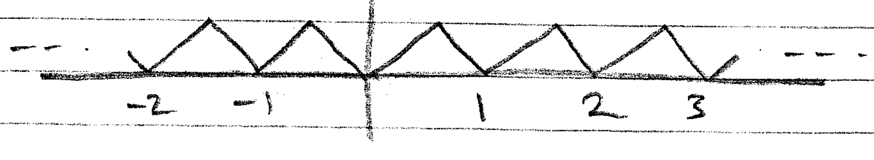
Fourier Convergence Theorem

Let f and f' be piecewise continuous on $[-l, l]$, where f is a $2l$-periodic function. Then the Fourier series of f converges to $f(x)$ at each point x at which f is continuous, and to the mean value

$$\frac{1}{2}\left(f(x^-) + f(x^+)\right)$$

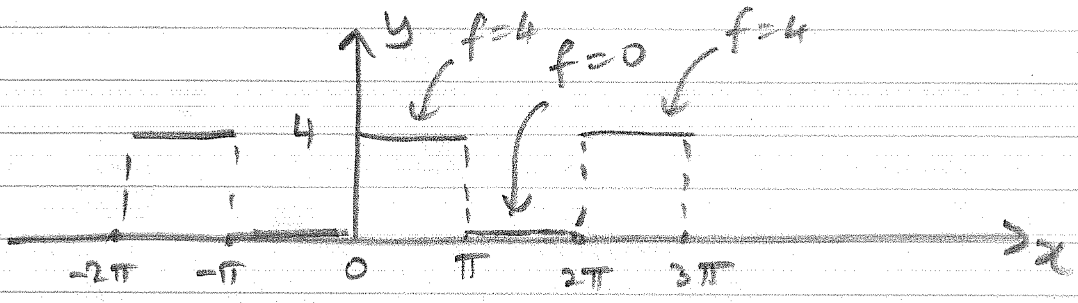
at each point at which f is not continuous.

Example (Saw-tooth)

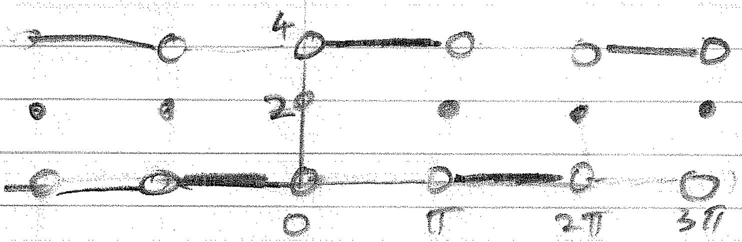


f and f' piecewise continuous. In fact, f is continuous so the Fourier series converges to $f(x)$.

Example Square wave



The Fourier series converges to



Lets calculate it:

$l = \pi$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} 4\,dx = 2$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} 4\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{4}{n}\sin nx\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} 4\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{4}{n}\cos nx\right]_0^{\pi} = \frac{-4}{n\pi}(\cos n\pi - 1)$$

$$= \frac{4}{n\pi}\left(1-(-1)^n\right)$$

So $$f(x) = 2 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

Make sure that you put myfourier.m in a folder and then
set the current directory of Matlab to be this folder.
Then just type the following into the command window.

```
>> x=linspace(-pi,pi);
>> plot(x,myfourier(x,7),x,myfourier(x,1000))
>>
```

Here's a listing of the Matlab file:

```
function y=myfourier(x,k)
%
% Calculates the sum of the first k terms of a Fourier series.
L = pi;
a0 = 2;
y = a0;

for n = 1:k
    an = 0;
    bn = 4*(1-(-1)^n)/(n*pi);
    y = y + an * cos(n*pi*x/L) + bn * sin(n*pi*x/L);
end
```