Separation of Variables (18.3)

Consider the diffusion problem with Dirichlet boundary conditions

$$u_t = \alpha^2 u_{xx} \qquad 0<x<L, \quad t>0$$
$$u(0,t) = u_1, \quad u(L,t) = u_2, \qquad t>0$$
$$u(x,0) = f(x) \qquad 0<x<L.$$

Here we assume that the end temperatures u_1 and u_2 are constant

Steady state solution

This is a solution of the PDE + boundary conditions that does not change with time. Can we find it?

Must have $u_t = 0$ so

$$u_{xx} = 0, \qquad u(0) = u_1, \quad u(L) = u_2$$

$$\Rightarrow u_x = c_1$$
$$\Rightarrow u = c_1 x + c_2$$

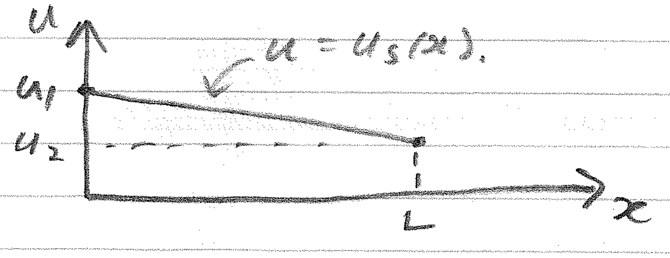
$\underline{x=0}$ $\quad u = c_2 \quad$ so $\quad c_2 = u_1.$

$\underline{x=L}$ $\quad u = c_1 L + c_2 = c_1 L + u_1 = u_2$

so $\quad c_1 = \dfrac{u_2 - u_1}{L}$

So the steady state solution is given by

$$u = u_s(x) = \left(\frac{u_2 - u_1}{L}\right)x + u_1$$



Physically we expect all solutions to tend towards the steady state solution as $t \to \infty$.

<u>Simplification of the boundary conditions</u>

Let $v(x,t) = u(x,t) - u_s(x)$

u and u_s are both solutions of a homogeneous linear PDE, so we know v is as well. v satisfies nicer b.c.

$$v(0,t) = u(0,t) - u_s(0) = u_1 - u_1 = 0$$
$$v(L,t) = u(L,t) - u_s(L) = u_2 - u_2 = 0$$

v satisfies a different initial condition:

$$\begin{aligned} v(x,0) &= u(x,0) - u_s(x) \\ &= f(x) - u_s(x) \end{aligned}$$

So we can find v by solving

$$\boxed{\begin{aligned} &v_t = \alpha^2 v_{xx} \\ &v(0,t) = 0, \quad v(L,t) = 0 \\ &v(x,0) = g(x) = f(x) - u_s(x). \end{aligned}} \quad *$$

<u>Note</u> The text takes a direct approach. It doesn't make use of u_s, which makes the solution a little messy.

<u>Solving * by separation of variables</u>

Try $u = X(x)T(t)$. This is a special form of solution.

$$u_t = X(x)T'(t) \qquad u_{xx} = X''(x)T(t)$$

Plug these into the PDE:

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Now divide each side by $X(x)T(t)$

$$\frac{T'(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)}$$

It turns out that the algebra will be simpler if we also divide by α^2

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}. \qquad \text{①}$$

(variables are separated!)

Notice that the left-hand-side is a function of t and the right-hand-side is a function of x. But x and t are independent variables; we can change one without changing the other. Thus equation ① makes sense only if each side = constant.

So $\dfrac{T'(t)}{\alpha^2 T(t)} = -k^2$, $\dfrac{X''(x)}{X(x)} = -k^2$

But $X''(x) = -k^2 X(x) \Rightarrow X(x) = A\cos kx + B\sin kx$,

$T'(t) = -k^2\alpha^2 T(t) \Rightarrow T(t) = Ce^{-k^2\alpha^2 t}$

Apply boundary conditions

Need $u(0,t) = u(L,t) = 0$. This works if $X(0)$ and $X(L) = 0$.

$X(0) = 0 \Rightarrow A = 0$ so $X(x) = B\sin(kx)$

$X(L) = 0 \Rightarrow B\sin(kL) = 0$. ②

If $B = 0$ then we end up with $u = 0$, which isn't much use. But if kL is a multiple of π then ② is satisfied.

So write $kL = n\pi$, $n = 1, 2, 3, \ldots$

or $k = \frac{n\pi}{L}$.

Our special solution is of the form

$$X(x)T(t) = CB \sin\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}$$

$$= K_n \sin\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}$$

$n = 1, 2, 3, \ldots$

But linear combinations of this will also satisfy the PDE and homogeneous boundary conditions. So go for an infinite linear combination:

$$U(x,t) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

<u>Finding the constants K_n</u>

Recall that $U(x,0) = g(x)$. But our series solution gives us $U(x,0)$ in terms of the constants:

$$U(x,0) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} = g(x) \qquad (3)$$

a known function

Can we find each K_n from this?
We make use of a special orthogonality property of sine functions:

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} \frac{L}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

We'll verify this below. But for now, multiply ③ by $\sin \frac{m\pi x}{L}$ and integrate each side from 0 to L

$$\int_0^L \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \int_0^L g(x) \sin \frac{m\pi x}{L}\, dx$$

If the series converges nicely, we can interchange the summation and integration so that

$$\sum_{n=1}^{\infty} K_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \int_0^L g(x) \sin \frac{m\pi x}{L}\, dx$$

But only one term in the sum on the LHS is non-zero; the term corresponding to $n = m$. So we obtain

$$K_m \frac{L}{2} = \int_0^L g(x) \sin \frac{m\pi x}{L}\, dx$$

and this gives a formula for K_m:

$$K_m = \frac{2}{L} \int_0^L g(x) \sin \frac{m\pi x}{L}\, dx$$

Example Suppose that $g(x) = T_0$ (a constant)

$$K_n = \frac{2}{L} \int_0^L T_0 \sin \frac{n\pi x}{2}\, dx = \frac{2T_0}{L} \left[\frac{-L}{n\pi} \cos n\pi x \right]_0^L$$

$$= \frac{2T_0}{n\pi} \left(-\cos n\pi + 1 \right)$$

$$= \frac{2T_0}{n\pi} \left(1 - (-1)^n \right)$$

$$= \begin{cases} \dfrac{4T_0}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

So $$u(x,t) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$= \frac{4T_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

i.e. $$u(x,t) = \frac{4T_0}{\pi}\left(\sin\frac{\pi x}{L}\, e^{-\left(\frac{\pi\alpha}{L}\right)^2 t} + \frac{1}{3}\sin\frac{3\pi x}{L}\, e^{-\left(\frac{3\pi\alpha}{L}\right)^2 t} + \frac{1}{5}\sin\frac{5\pi x}{L}\, e^{-\left(\frac{5\pi\alpha}{L}\right)^2 t} + \cdots \right)$$

We could compute the solution using Matlab.

For the case of a 10 cm copper rod, initially put in boiling water and removed at $t=0$ we have

$L = 10$, $\alpha^2 = 1.14\ \text{cm}^2/\text{sec}$, $T_0 = 100$.

The Matlab code is shown on the next page.

We can plot a graph showing u at several time values:

```
x = linspace(0,10);
plot(x, u(x,0), x, u(x,1), x, u(x,5), x, u(x,12))
```

The graph is shown after the next page.

<u>Verifying orthogonality condition</u>

We use the trig identity

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L \cos(n-m)\frac{\pi x}{L} - \cos(n+m)\frac{\pi x}{L}\,dx$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin(n-m)\frac{\pi x}{L} - \frac{L}{(n+m)\pi}\sin(n+m)\frac{\pi x}{L}\right]_0^L \quad \text{if } n \neq m$$

$= 0$ if $n \neq m$

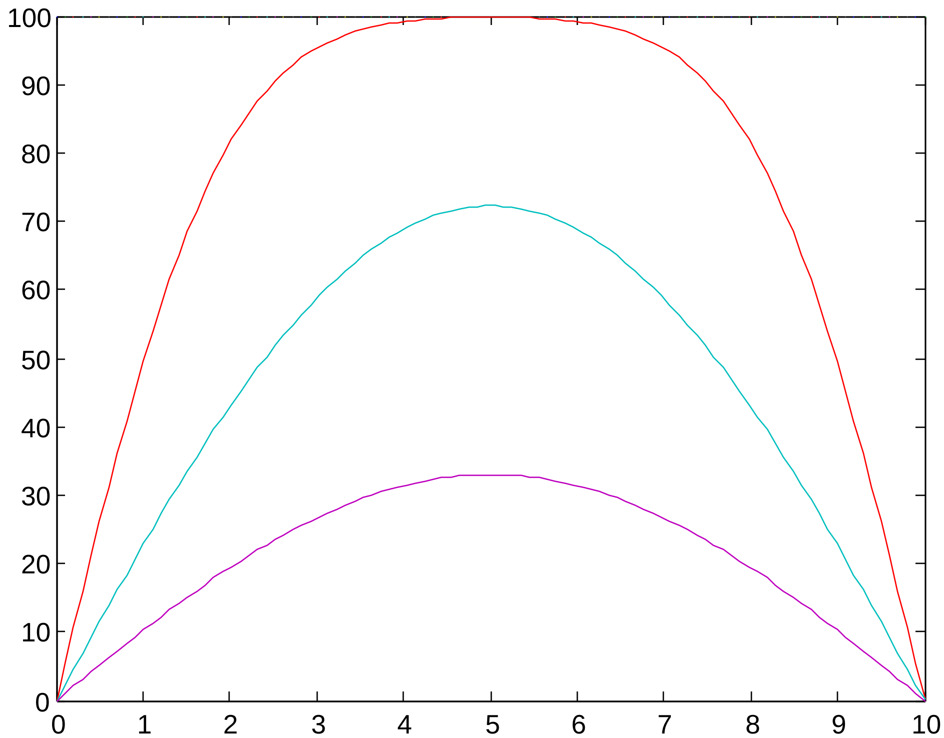
But if $n=m$,

$$\int_0^L \sin^2\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_0^L 1-\cos\frac{2n\pi x}{L}\,dx$$

$$= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_0^L$$

$$= \frac{L}{2}.$$

```
function y=U(x,t)
%
% Calculates solution of heat equation
L = 10;
alpha = sqrt(1.14); % for copper
T0 = 100;
infinity=100; % a large number!
if t==0
    y=T0;
else
    y=0;
    for n=1:infinity
        y=y +((1-(-1)^n)/n)*sin(n*pi*x/L)*exp(-(n*pi*alpha/L)^2*t);
    end
    y=y*2*T0/pi;
end
```



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