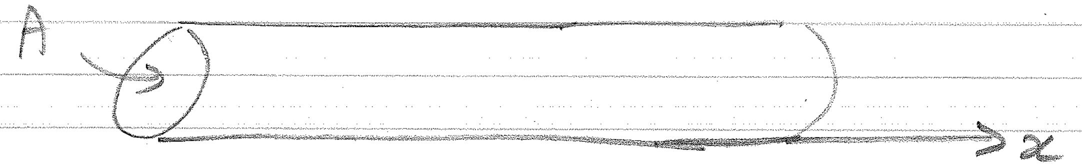
Physical Derivation of the diffusion equation for heat conduction 18.2.3

Consider a uniform rod of cross-sectional area A.



We derive the heat equation using a conservation of energy argument. Conservation laws often lead to PDEs, so this is a typical approach.

Let e be the energy per unit volume in the rod. e may be a function of x and t.

The energy in a thin slice between x and $x + dx$ is $e \underbrace{A\,dx}_{\text{volume}}$.

The energy between x_1 and x_2 is

$$A \int_{x_1}^{x_2} e(x,t)\,dx$$

The rate of change of energy in this segment is

$$\frac{d}{dt} A \int_{x_1}^{x_2} e(x,t)\,dx = A \int_{x_1}^{x_2} \frac{\partial e}{\partial t}\,dx$$

The heat flux ϕ at a distance x along the rod is the rate at which heat energy passes to the right at x per unit area.

The rate of change of energy between x_1 and x_2 is thus

$$(\text{rate in}) - (\text{rate out})$$

$$= A\,\phi(x_1, t) - A\,\phi(x_2, t)$$

Hence we must have

$$\int_{x_1}^{x_2} \frac{\partial e}{\partial t}\, dx = \phi(x_1, t) - \phi(x_2, t)$$

This is an example of an integral conservation law.

Differentiate wrt x_2

$$\frac{\partial e}{\partial t}(x_2, t) = -\frac{\partial \phi}{\partial x}(x_2, t)$$

But x_2 is arbitrary so we can just write

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} = 0$$

<u>Relationships between e, ϕ and temperature</u>

It turns out that the heat energy can be approximated by a linear function of the temperature u

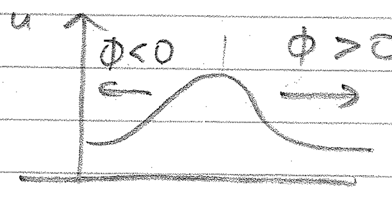
$$e = \sigma c u + \text{constant}$$

The proportionality "constant" is σc
where σ = mass density of rod's material
c = specific heat of rod's material

Hence $\dfrac{\partial e}{\partial t} = \sigma c \dfrac{\partial u}{\partial t}$

Fourier realised that ϕ should be positive if $\dfrac{\partial u}{\partial x} < 0$ and negative if $\dfrac{\partial u}{\partial x} > 0$

so he postulated $\phi = -k \dfrac{\partial u}{\partial x}$



This turned out surprisingly accurate.

k is called the <u>thermal conductivity</u>

So $\dfrac{\partial e}{\partial t} + \dfrac{\partial \phi}{\partial x} = 0$

gives $\delta C \dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} = 0$

so $\dfrac{\partial u}{\partial t} = \dfrac{k}{\delta C} \dfrac{\partial^2 u}{\partial x^2} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$

where $\alpha^2 = \dfrac{k}{\delta C}$ is called the thermal diffusivity

Note the text also considers heat loss along the bar + moving bar.

Initial Condition

The heat equation is first order in time so only one initial condition is needed:

$$u(x, 0) = f(x), \quad 0 < x < L.$$

This gives the temperature distribution in the rod at time $t = 0$.

Boundary conditions

Suppose that the ends of the rod lose heat to the environment according to Newton's Law of Cooling which states that the rate of heat loss from an object is proportional to the difference between the object's temperature and the environment's temperature.

If the environment near $x = L$ is at temperature u_2 this gives

$$-k \frac{\partial u}{\partial x}(L, t) = h\,(u(L,t) - u_2)$$

i.e. $\dfrac{\partial u}{\partial x}(L, t) + \dfrac{h}{k} u(L, t) = \dfrac{h}{k} u_2 \qquad (1)$

Similarly, if the environment near $x=0$ is at temperature u_1 then we have

$$k \frac{\partial u}{\partial x}(0,t) = h(u(0,t) - u_1)$$

so $$\frac{\partial u}{\partial x}(0,t) - \frac{h}{k} u(0,t) = -\frac{h}{k} u_1 \quad — ②$$

h is called a heat transfer coefficient.
① and ② are called Robin boundary conditions.

Extreme cases:

(a) $h = 0$ (perfect insulation).
Then ① and ② become

$\frac{\partial u}{\partial x} = 0$ at $x = 0, L$. These are called Neumann boundary conditions

(b) $h \to \infty$ (perfect heat transfer between ends of rod and the environment).

First divide ① by h

$$\frac{1}{h} \frac{\partial u}{\partial x}(L,t) + \frac{1}{k} u(L,t) = \frac{1}{k} u_2$$

Now let $h \to \infty$. The boundary condition becomes

$$u(L,t) = u_2$$

Similarly, the b.c. in ② would become

$$u(0,t) = u_1$$

These are called Dirichlet boundary conditions.