Maths 361 Lecture Notes

Text: 18.1, 18.2

1. PDEs (partial differential equations)

In this course we study PDEs, examples of which are

(a) $$\frac{\partial T}{\partial t} = \alpha^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

This is called the heat equation or diffusion equation. It tells us how the temperature $T(x, y, z, t)$ at a point (x, y, z) in a material changes with time t. α^2 is called the heat diffusivity. Sometimes we write k instead of α^2 but the square emphasises the fact that the diffusivity is positive.

The diffusion equation also models the diffusion of particles in a gas or liquid. In this case $T(x, y, z, t)$ would be the concentration of particles.

We also write the diffusion equation as

$$\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$$

where ∇^2 ("del squared") is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Sometimes T depends on only a few of the independent variables x, y, z, t

e.g. $$\frac{\partial T}{\partial t} = \alpha^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \qquad \text{2 D diffusion eq.}$$

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2} \qquad \text{1 D diffusion eq.}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{(steady state, no } t \text{ dependence}$$

(Laplace's equation).

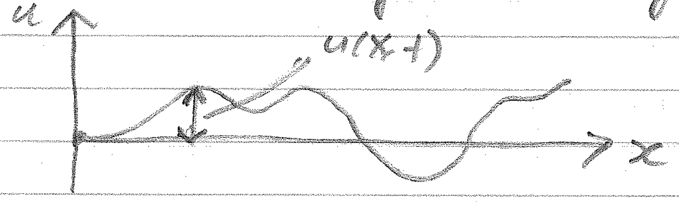
(b) The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Models the motion of waves through a medium. $c =$ wave speed.

In 1-D $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

models the motion of a string.



It also models 1-D sound waves (e.g. in a pipe)

In 3-D the equation models sound waves and electromagnetic waves (light, radio, x-rays, etc.)

(c) Laplace's equation $\nabla^2 u = 0$

Arises as a steady state of the diffusion and wave equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (3D)$$

The 1-D version is easy to solve!

$$\frac{d^2 u}{dx^2} = 0 \Rightarrow \frac{du}{dx} = c_1 \Rightarrow u = c_1 x + c_2$$

In this course we discuss these PDEs and learn how to solve them using a technique known as "separation of variables".

Preliminary Concepts

A differential equation is a partial differential equation if it contains partial derivatives of the dependent variable.

e.g. $$\frac{\partial T}{\partial t} = \alpha^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

T = dependent variable (temperature)
x, y, z, t = independent variables (here space coordinates and time).

The order of a PDE is the order of the highest derivative in the PDE

So the heat, wave and Laplace equations are of order 2.

$$\rho \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = 0$$ is of order 4

(This models the vibrations of a rod).

Operator Notation

Instead of writing a PDE completely

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = \sin(x)$$

we might write $Lu = f$ where

$$L = \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}$$

denotes a differential operator. L acts on certain functions. When defining a differential operator we should specify the set of functions on which it acts

e.g. functions u of x, t defined for $0 < x < 1$, $t > 0$ for which $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ are continuous.

A differential operator L is <u>linear</u> if

$$L(\alpha u + \beta v) = \alpha L u + \beta L v$$

for all functions u and v and all constants α and β.

e.g. $L = \frac{\partial}{\partial t} + x \frac{\partial^2}{\partial x^2}$

is linear. If u and v are functions of x and t then

$$L(\alpha u + \beta v) = \left(\frac{\partial}{\partial t} + x\frac{\partial^2}{\partial x^2}\right)(\alpha u(x,t) + \beta v(x,t))$$

$$= \frac{\partial}{\partial t}(\alpha u(x,t) + \beta v(x,t))$$

$$+ x\frac{\partial^2}{\partial x^2}(\alpha u(x,t) + \beta v(x,t))$$

$$= \alpha\frac{\partial u}{\partial t} + \beta\frac{\partial v}{\partial t} + x\alpha\frac{\partial^2 u}{\partial x^2} + x\beta\frac{\partial^2 v}{\partial x^2}$$

$$= \alpha L u + \beta L v$$

e.g. $Lu = u\frac{\partial u}{\partial t}$ is not linear

Just take $v = 0$

$$L(\alpha u) = \alpha u \frac{\partial(\alpha u)}{\partial t}$$

$$= \alpha^2 u \frac{\partial u}{\partial t} \neq \alpha L u \quad \text{unless } \alpha^2 = \alpha.$$

<u>Note</u> If L is linear then

$$L(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_N u_N) = \alpha_1 L u_1 + \alpha_2 L u_2 + \cdots + \alpha_N L u_N$$

A <u>linear</u> <u>PDE</u> is one that can be written

$Lu = f$ where L is linear.

If $f = 0$ the PDE is called "homogeneous"
If $f \neq 0$ " " " "nonhomogeneous".

Suppose that v is a solution of a linear PDE

$$Lu = f$$

and that $u_1, u_2, u_3 \ldots$ are solutions of the homogeneous linear PDE

$$Lu = 0$$

Then $u = \sum_{j=1}^{N} c_j u_j + v$

is a solution of $Lu = f$

<u>check</u> $Lu = L\left(\sum_{j=1}^{N} c_j u_j + v\right)$

$$= \sum_{j=1}^{N} c_j Lu_j + Lv$$

$$= \sum_{j=1}^{N} 0 + f$$

$$= f$$

In fact we will often be able to get an infinite number of solutions to $Lu = 0$ and find general solutions to $Lu = f$ via

$$u = \sum_{j=1}^{\infty} c_j u_j + v$$

e.g. diffusion equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$

for $0 < x < 1$, $t > 0$. For each integer $j \geq 1$,

$$u_j(x,t) = \sin(j\pi x)e^{-j^2\pi^2 t}$$

[check this]

is a solution so we should be able to construct other solutions using

$$u(x,t) = \sum_{j=1}^{\infty} c_j \sin(j\pi x)e^{-j^2\pi^2 t}$$