Sturm - Liouville BVPs

Up until now mostly seen just $\frac{d^2\varphi}{dx^2} = -\lambda^2\varphi$ as the ODE to give the eigenfunctions. But usually not so simple.

Eg

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(x) \frac{\partial u}{\partial x} \right) + \alpha u$$

varying diffusion coefficient

heat source

Sep. variables

$$u = \varphi(x) T(t) \Rightarrow \varphi T' = \frac{\partial}{\partial x}(D\varphi' T) + \alpha \varphi T$$

$$\Rightarrow \varphi T' = T (D\varphi')' + \alpha\varphi T$$

$$\Rightarrow \frac{T'}{T} = \frac{(D\varphi')'}{\varphi} + \alpha = -\lambda$$

Don't know sign of λ yet.

So, the eqn for φ is

$$(D\varphi')' + (\alpha + \lambda)\varphi = 0$$

Note: $D = D(x)$ & maybe $\alpha = \alpha(x)$ also. So a lot more complicated than before.

Questions: – How do we know our "Fourier" expansions still work? (Not just sines & cosines now)

– i.e. are eigenfunctions still a complete orthogonal set?

Such des are called Sturm-Liouville eqs.

<u>Classification</u>

The Sturm-Liouville o.d.e. is.

$$\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) + q(x)\varphi + \lambda\varphi = 0$$

$$\uparrow$$

or $\lambda\sigma(x)\varphi$

Different kinds of B.C.s.

$$\begin{cases} \varphi = 0 \quad \text{at} \quad x = 0, L & \text{(Dirichlet)} \\ \dfrac{d\varphi}{dx} = 0 \quad \text{at} \quad x = 0, L & \text{(Neumann)} \\ \dfrac{d\varphi}{dx} = \pm h\varphi \quad \text{at} \quad x = 0, L & \text{(Robin condition)} \\ \varphi(-L) = \varphi(L) & \\ \varphi'(-L) = \varphi'(L) & \text{(Periodic)} \end{cases}$$

Can also get combinations of these basic types.

<u>Regular problem</u>

$$\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) + q(x)\varphi + \lambda\varphi = 0, \quad a < x < b$$

$$\text{B.C.s} \begin{cases} \beta_1\varphi(a) + \beta_2\dfrac{d\varphi}{dx}(a) = 0 \\ \beta_3\varphi(b) + \beta_4\dfrac{d\varphi}{dx}(b) = 0 \end{cases}$$

Note: linear & unmixed

p, q, σ real & continuous everywhere, $p > 0$ & $\sigma > 0$

Important properties

1) All eigenvalues are real

2) $\exists$ infinite number of e-values

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$$

So a) there is a smallest, λ_1

b) there is no largest, & $\lambda_n \to \infty$ as $n \to \infty$

3) For each e-value λ_n, $\exists$ e-function $\varphi_n(x)$, unique up to multiplicative constant. $\varphi_n(x)$ has $n-1$ zeros in (a,b)

4) φ_ns form a complete set. Any $f \in PS(a,b)$ can be written $f(x) \sim \sum_{n=1}^{\infty} a_n \varphi_n$, which converges to $\dfrac{f(x^+) + f(x^-)}{2}$

5) $\int \varphi_n \varphi_m \, dx = 0$ if $n \neq m$

6) Rayleigh Quotient:

$$\lambda = \frac{-p\varphi \dfrac{d\varphi}{dx}\Big|_a^b + \displaystyle\int_a^b \left[p\left(\frac{d\varphi}{dx}\right)^2 - q\varphi^2 \right] dx}{\displaystyle\int_a^b \varphi^2 \sigma \, dx}$$

Proofs

A few later. None in great detail.

Eg $\qquad \dfrac{d^2\varphi}{dx^2} + \lambda\varphi = 0 \quad , \quad \varphi(0) = \varphi(L) = 0$

(Seen this many times already)

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n = \sin\frac{n\pi x}{L}$$

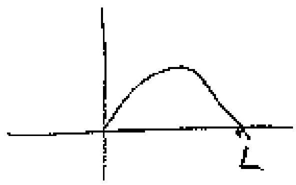
Note:

1) all real (if we had looked, found no complex ones)

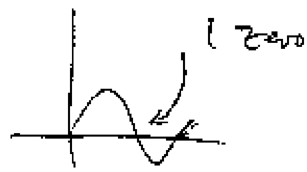
2) clearly ordered

3) $\varphi_n = \sin\dfrac{n\pi x}{L}$ is unique. What about numbers of zeros?

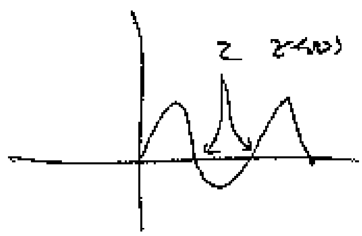
n=1



n=2



n=3



Clearly the result follows.

4) We already know that $f \in PS(a,b) \Rightarrow f \sim \sum a_n\varphi_n$ (Fourier)

5) Already know φ_ns are orthogonal.

6) $p(x)=1 \ , \ q(x)=0 \ , \ u(0)=u(L)=0$

$$\Rightarrow \quad \lambda_n = \int_0^L (\varphi_n')^2\,dx \Big/ \int_0^L \varphi_n^2\,dx$$

$$= \frac{n^2\pi^2}{L^2}\int_0^L \cos^2\frac{n\pi x}{L} \Big/ \int_0^L \sin^2\frac{n\pi x}{L} = \frac{n^2\pi^2}{L^2} \text{ as req'd.}$$

Eg Heat flow in non-uniform rod:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(D(x)\,\dfrac{\partial u}{\partial x}\right) & \\ u(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0 & \text{B.C.} \\ u(x,0) = f(x) & \text{I.C.} \end{cases}$$

Separate variables: $u(x,t) = \varphi(x)\,T(t)$

$$-\lambda = \varphi T' = T\,(D\varphi')' \Rightarrow \frac{T'}{T} = -\lambda$$

and $$\begin{cases} (D\varphi')' = -\lambda\varphi \quad \longleftarrow \text{not always easy to solve.} \\ \left[D(x)\varphi'' + D'(x)\varphi'(x) + \lambda\varphi(x) = 0 \right] \\ \varphi(0) = 0 \\ \varphi'(L) = 0 \end{cases}$$

T is easy $T(t) = c\,e^{-\lambda_n t}$, for e-value λ_n which we don't know yet

But, the SL theorems tell us that the λ_ns and φ_ns exist and are complete and orthogonal. So we can just write everything in terms of λ_n & φ_n.

A soln. is $\varphi_n(x)\,e^{-\lambda_n t}$, so look for all solns using superposition

i.e. $$u(x,t) = \sum_{n=1}^{\infty} a_n \varphi_n(x)\,e^{-\lambda_n t}$$

Now use orthogonality to determine a_n

$$u(x,0) = f(x) = \sum a_n \varphi_n(x)$$

$$\Rightarrow \quad \int f \varphi_m = \sum a_n \int \varphi_n \varphi_m$$

$$\Rightarrow \quad a_m = \frac{\int_0^L f(x)\,\varphi_m(x)}{\int_0^L \varphi_m^2(x)\,dx}$$ as expected.

This completely characterises the soln, even though we don't actually know the λ_n & φ_n.

<u>Long-time behaviour</u> As $t \to \infty$, $u(x,t) \to a_1 \varphi_1 e^{-\lambda_1 t}$ since all the other $\lambda_n > \lambda_1$. But is $\lambda_1 > 0$ (giving decay) or is $\lambda_1 < 0$ (giving blowup). Physically, we expect $\lambda_1 > 0$ of course.

From the Rayleigh Quotient

$p(x)$ $(q=0)$

$$\lambda_n = \underbrace{0}_{\text{Boundary terms are zero}} + \frac{\int_0^L D(x)\left(\frac{d\varphi_n}{dx}\right)^2}{\|\varphi_n\|^2} \geq 0$$

since $D(x) \geq 0$ by physical definition. thus the solution must decay.

Time to prove some results: (Not all of them. Completeness is too difficult, for instance)

Use linear operator notation

$$L(u) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u$$

Just a shorthand. So, SL eqn is

$$L(\varphi) + \lambda\varphi = 0 \quad , \quad \text{with B.C. also.}$$

(Note the similarity to matrix notation $Ax = \lambda x$)

<u>Lagrange's identity</u>

$$uL(v) - vL(u) = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)$$

Proof by simple substitution.

<u>Green's formula</u>

$$\int_a^b uL(v) - vL(u) = p\left(uv' - vu'\right)\Big|_a^b$$

<u>Eg</u> $p=1$ & $q=0$ $\Rightarrow$ $L = \frac{d^2}{dx^2}$ $\Rightarrow$

$$\int uv'' - vu'' = \frac{d}{dx}\left(uv' - vu'\right).$$

Easily checked.

//.

Another special case is where

$$p(uv' - vu')\Big|_a^b = 0 \quad \text{because of the B.C.s.}$$

Then $\int [uL(v) - vL(u)] = 0$. Very useful.

Eg. Spse the B.C.s are

$$\left.\begin{array}{l} \varphi(a) = 0 \\ \varphi'(b) + h\varphi(b) = 0 \end{array}\right\}$$

Then both u & v satisfy these conditions, so

$$u(a) = 0 \qquad v(a) = 0$$
$$u'(b) + hu(b) = 0 \qquad v'(b) + hv(b) = 0$$

Then

$$\begin{aligned}(uv' - vu')\Big|_a^b &= u(b)v'(b) - v(b)u'(b) \\ &\quad - u(a)v'(a) + v(a)u'(a) \\ &= u(b)v'(b) - v(b)u'(b), \quad \text{since } u(a) = v(a) = 0 \\ &= u(b)(-hv(b)) - v(b)(-hu(b)) \\ &= 0.\end{aligned}$$

So for these common B.C.s, the boundary terms in Lagrange's identity vanish //

In fact:

<u>Result</u>. If u & v both satisfy the same set of homogeneous regular SL Boundary conditions, then
$$\int_a^b \left(uL(v) - vL(u)\right) = 0$$
//

In this case, L (with the appropriate B.Cs) is called <u>self-adjoint</u>.

<u>Eg</u> If the BCs are
$$\varphi(a) = \varphi(b)$$
$$p(a)\varphi'(a) = p(b)\varphi'(b)$$

Show that L (with those BCs) is self-adjoint.
Only have to check
$$p(uv' - vu')\Big|_a^b = p(b)\left(u(b)v'(b) - v(b)u'(b)\right)$$
$$- p(a)\left(u(a)v'(a) - v(a)u'(a)\right)$$
$$= p(b)\left(u(b)v'(b) - v(b)u'(b)\right)$$
$$- \underset{u(b)}{u(a)}\,p(b)v'(b) + \underset{v(b)}{v(a)}\,p(b)u'(b)$$
$$= 0$$
as req'd.
//

Orthogonality

We now prove that eigenfunctions are orthogonal

Spse we have 2 e-fns φ_n & φ_m

$$\begin{cases} L(\varphi_n) + \lambda_n \varphi_n = 0 \\ L(\varphi_m) + \lambda_m \varphi_m = 0 \end{cases}$$

Apply Green's formula

$$\int \varphi_m L(\varphi_n) - \varphi_n L(\varphi_m) = p(x)\left(\varphi_m \varphi_n' - \varphi_n \varphi_m'\right)\Big|_a^b$$

Now use $L(\varphi_n) = -\lambda_n \varphi_n$ etc. to get

$$(\lambda_m - \lambda_n) \int \varphi_n \varphi_m = p(x)(\varphi_m \varphi_n' - \varphi_n \varphi_m')\Big|_a^b$$

From an above result we know that for homog. B.C. of regular SL type the RHS $= 0$

i.e.
$$(\lambda_m - \lambda_n) \int \varphi_n \varphi_m = 0$$

Since $\lambda_n \neq \lambda_m$ by assumption, it follows that

$$\int \varphi_n \varphi_m = 0 \quad \text{as required.} \quad //$$

Eigenvalues are real.

Spse λ was a complex e-value

$$L(\varphi_n) + \lambda \varphi_n = 0$$

Take complex conjugate

$$\overline{L(\varphi_n)} + \bar{\lambda}_n \bar{\varphi}_n = 0$$

conjugate goes inside the differentiation $\longleftrightarrow$

$$L(\bar{\varphi}_n) + \bar{\lambda}_n \bar{\varphi}_n = 0$$

Apply Greens theorem to φ_n and $\bar{\varphi}_n$ to get

$$(\lambda_n - \bar{\lambda}_n) \underbrace{\int \varphi_n \bar{\varphi}_n}_{\|\varphi_n\|^2} = 0$$

$\Rightarrow$ $\lambda_n = \bar{\lambda}_n$ since $\|\varphi_n\|^2 \neq 0$ by definition. //

Rayleigh Quotient.

Start from

$$\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) + p(x)\varphi + \lambda\varphi = 0 \qquad (\text{i.e. } L(\varphi) + \lambda\varphi = 0)$$

Multiply by φ and integrate

$$\int_a^b \varphi \frac{d}{dx}(p\varphi') + q\varphi^2 + \lambda\varphi^2 = 0$$

Now just solve for λ.

$$\lambda = \frac{-\left\{\int_a^b \left(\varphi \frac{d}{dx}(p\varphi') + q\varphi^2\right)dx\right\}}{\int_a^b \varphi^2}$$

Now integrate by parts $\quad \int \varphi\,(p\varphi')' = \varphi p \varphi'\Big|_a^b - \int p\varphi'^2$

So
$$\lambda = \frac{-p\varphi\varphi'\Big|_a^b + \int_a^b \left(p(\varphi')^2 - q\varphi^2\right)dx}{\int_a^b \varphi^2}$$

as required //

This can give useful conditions on λ.

Eg. If $q \leq 0$ and $p\varphi\varphi'\Big|_a^b \leq 0$, we must have $\lambda \geq 0$.

Remember that commonly $p\varphi\varphi'\Big|_a^b = 0$.

In heat flow, $q < 0 \Rightarrow$ loss of heat during PDE

$\Rightarrow$ decaying solutions (i.e. $\lambda \geq 0$)

So this makes good physical sense.

Minimisation Principle

$$\lambda_1 = \min \left\{ \frac{-pu\frac{du}{dx}\Big|_a^b + \int_a^b \left(p(u')^2 - qu^2\right)}{\int_a^b u^2} \right\}$$

where the minimisation is over all continuous fns that satisfy the B.C.s. At the min (attained at u_1, say) u_1 is the eigenfunction.

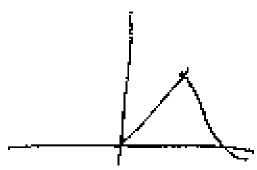
Can use this to get upper bounds on λ_1, just by plugging in continuous functions that obey the B.C.

Eg:
$$\left.\begin{array}{l} \varphi'' + \lambda\varphi = 0 \\ \varphi(0) = \varphi(1) = 0 \end{array}\right\} \quad \lambda_n = n^2\pi^2, \quad \varphi_n = \sin n\pi$$

our old friend.

For $n=1$, $\lambda_1 = \pi^2$. Approximate π^2 by evaluating that nasty mess for other functions that are $=0$ at $x=0,1$.

Try $u = \begin{cases} x, & x < \frac{1}{2} \\ 1-x, & x > \frac{1}{2} \end{cases}$



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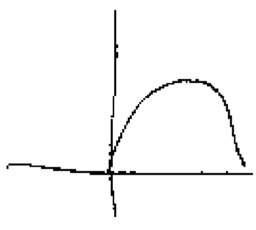
This gives $\lambda_1 \leq 12$ (details as exercise)

Or $u = x - x^2$



$\Rightarrow \lambda_1 \leq 10$.

Second estimate is more accurate.

Example of other B.Cs.

Heat flow
$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(L,t) = -h\, u(L,t) \end{cases}$$

Newton's Law of Cooling at this end.

$$u(x,t) = \varphi(x)\, T(t)$$

So $T' = -\lambda T$ and.

$$\begin{cases} \varphi'' + \lambda\varphi = 0 \\ \varphi(0) = 0 \\ \varphi'(L) + h\varphi(L) = 0 \end{cases}$$

This is trickier.

We now have to worry about $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

General solution is

$$\varphi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}\, x$$

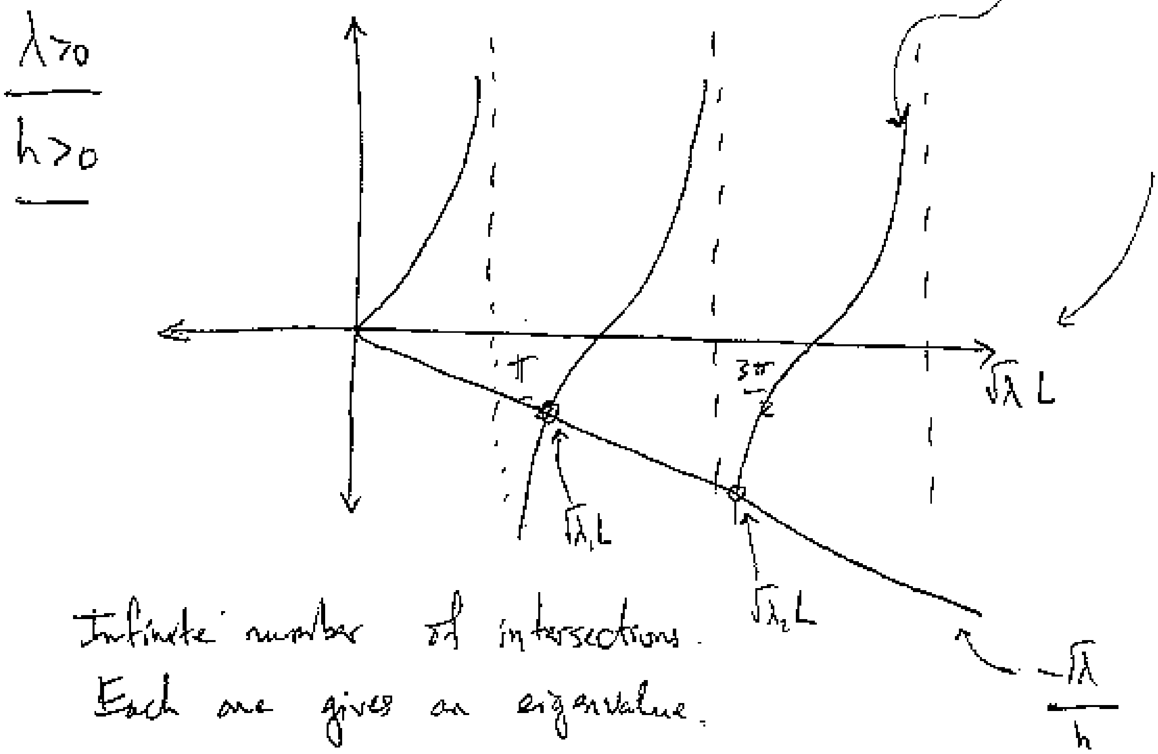
$$\varphi(0) = 0 \Rightarrow c_1 = 0 \Rightarrow \varphi = c_2 \sin\sqrt{\lambda}\, x$$

$$\varphi'(L) + h\varphi(L) = 0 \Rightarrow c_2\sqrt{\lambda}\cos\sqrt{\lambda}L + h\, c_2 \sin\sqrt{\lambda}L = 0$$

$$\Rightarrow \boxed{\tan\sqrt{\lambda}L = -\frac{\sqrt{\lambda}}{h}}$$

We have to solve this. Nasty. Can't solve exactly

Find eigenvalues approximately by sketching



Infinite number of intersections. Each one gives an eigenvalue.

Note: - Don't count $\lambda = 0$ ($\lambda > 0$ only for now)

- As expected, a lowest λ and an infinite, unbounded number of evalues

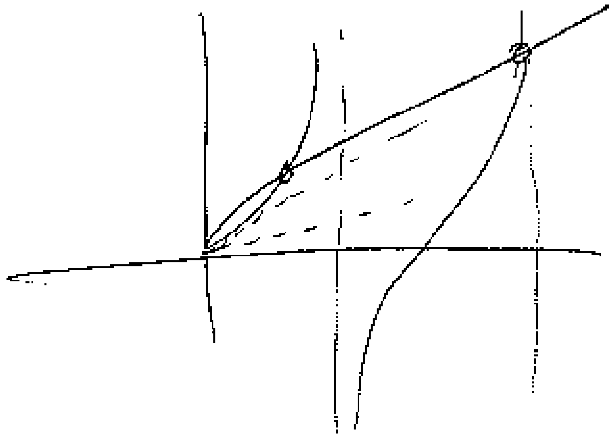
Also $\frac{\pi}{2} < \sqrt{\lambda_1}L < \pi$

$\frac{3\pi}{2} < \sqrt{\lambda_2}L < 2\pi$

and as $n \to \infty$, $\sqrt{\lambda_n}L \to \frac{(2n-1)\pi}{2}$

In practise, find the roots numerically using Newton's method or similar.

$\underline{\lambda>0}$
$\underline{h<0}$ Unphysical, because of heat input at one end.



Similar analysis
three case
i) $-1 < hL < 0$
ii) $hL = -1$
iii) $hL < -1$

$\underline{\lambda=0}$

Is $\lambda=0$ an e-value?

$$\frac{d^2\varphi}{dx^2} = 0 \quad \Rightarrow \quad \varphi = C_1 + C_2 x$$

$$\varphi(0) = 0 \quad \Rightarrow \quad \varphi = C_2 x$$

$$\varphi'(L) + h\varphi(L) = 0 \quad \Rightarrow \quad C_2 + hC_2L = 0$$

$$\Rightarrow \quad C_2(1+hL) = 0 \quad \Rightarrow \quad C_2 = 0$$

if $h>0$

However, if $hL = -1$, then $\lambda=0$ is a e-value, with eigenfunction x.

$\underline{\lambda<0}$ Any negative evalues?

$$\frac{d^2\varphi}{dx^2} = s\varphi \quad , \quad s>0$$

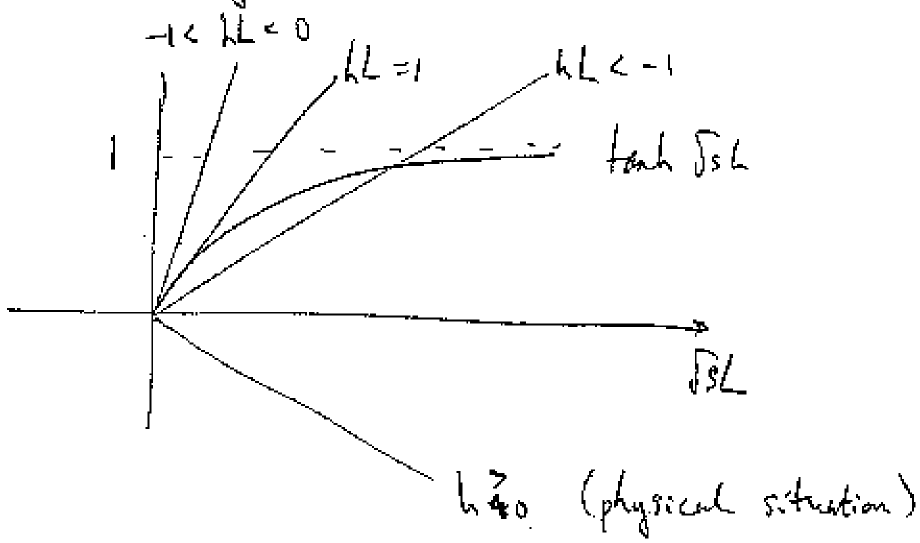
Soln is $\varphi = C_1 \cosh\sqrt{s}\,x + C_2 \sinh\sqrt{s}\,x$

$$\varphi(0) = 0 \Rightarrow c_1 = 0$$

So $\varphi = c_2 \sinh \sqrt{s}\,x$

$$\varphi'(L) + h\varphi(L) = 0 \Rightarrow \tanh \sqrt{s} L = -\frac{\sqrt{s}}{h}$$

Graphical soln again: Remember: $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$



So, $h > 0 \Rightarrow$ no negative evalues (no solutions for s).

If $h < 0$ then we either have 1 or 0 negative e-values

<u>Summary of eigenfunction</u>

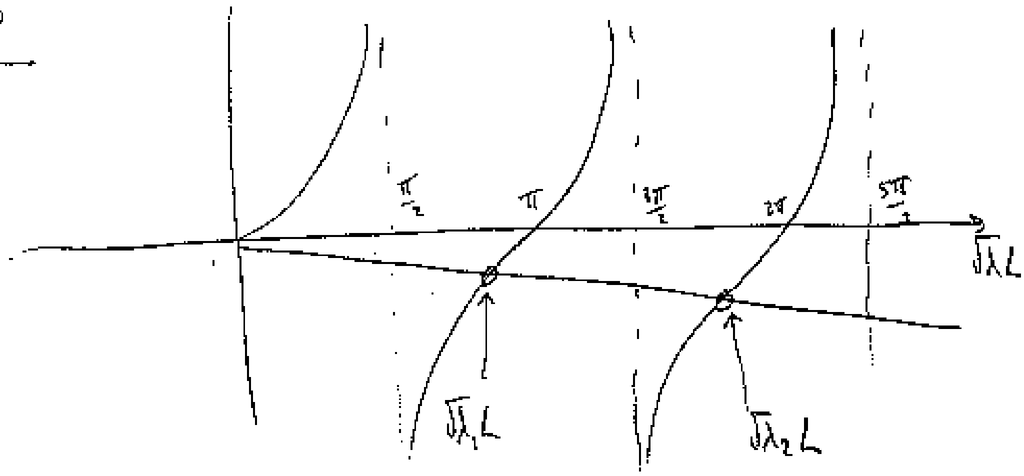
		$\lambda > 0$	$\lambda = 0$	$\lambda < 0$
Physical	$h > 0$	$\sin \sqrt{\lambda} x$		
	$h = 0$	$\sin \sqrt{\lambda} x$		
Non-physical	$-1 < hL < 0$	$\sin \sqrt{\lambda} x$		
	$hL = -1$	$\sin \sqrt{\lambda} x$	X	
	$hL < -1$	$\sin \sqrt{\lambda} x$		$\sinh \sqrt{s_1} x$

$s_1 = -\lambda_1$, lowest e-value.

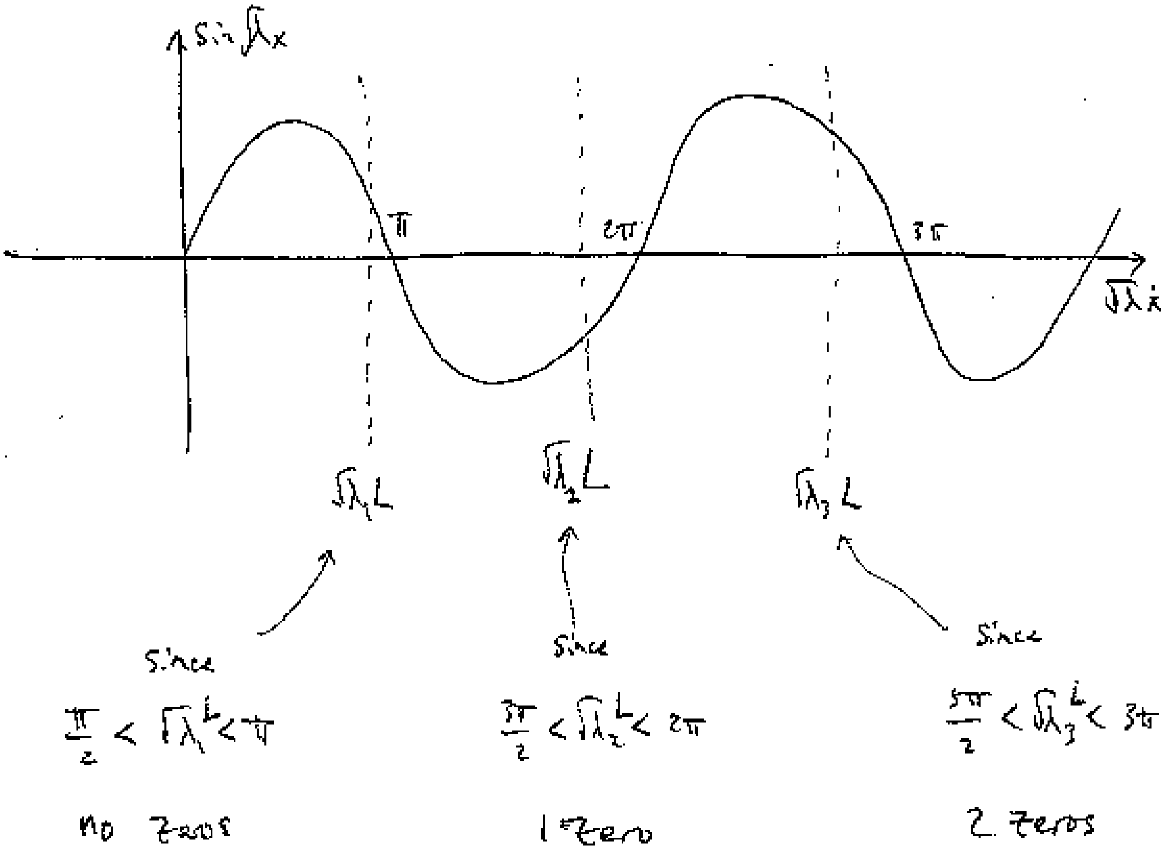
Zeros of eigenfunctions

Recall: $\varphi_n(x)$ has $n-1$ zeros in $(0, L)$. We consider only ~~two~~ one cases in the previous example.

$h > 0$ & $\lambda > 0$



Eigenfunction is $\sin(\sqrt{\lambda}\,x)$. Determine zeros by plotting $\sin\sqrt{\lambda}x$, measuring at length $x=L$, and counting zeros.



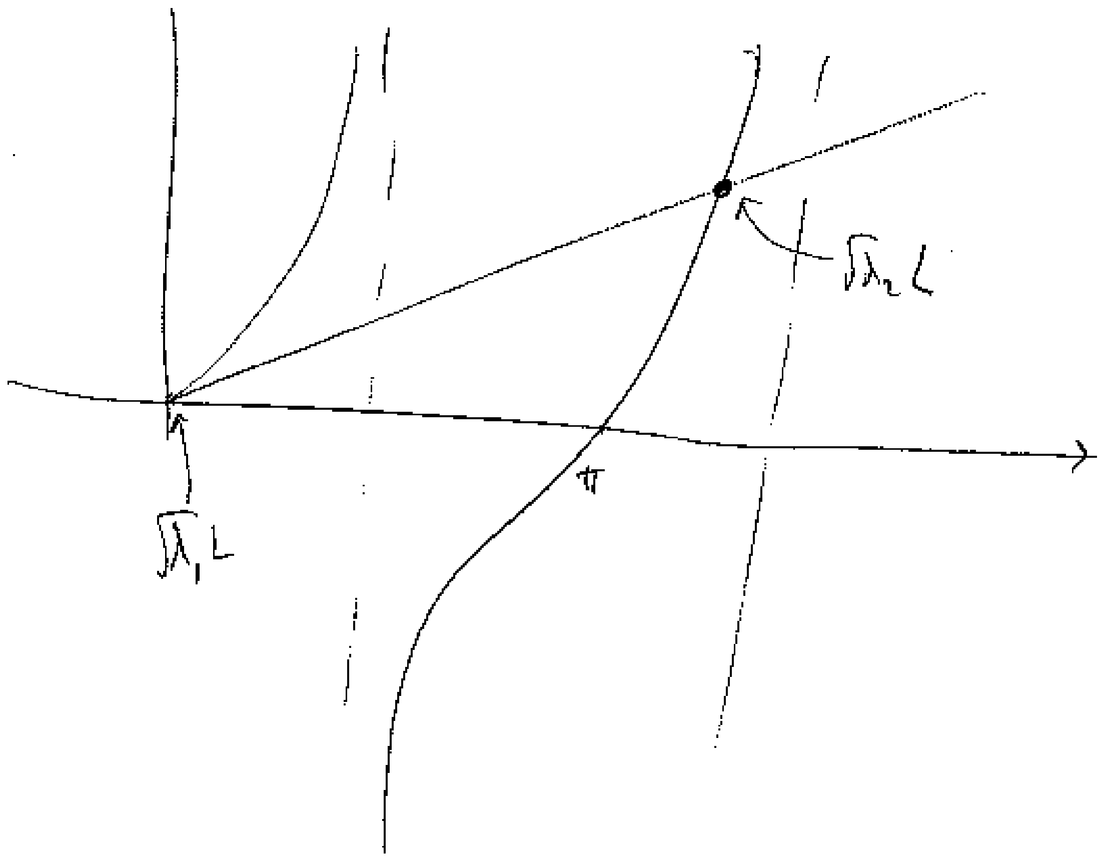
What about

$-1 = hL$

Eigenfunctions are

$$\varphi_1 = x$$

$\varphi_2 = \sin\sqrt{\lambda_2}x$ where $\sqrt{\lambda_2}L$ is given by



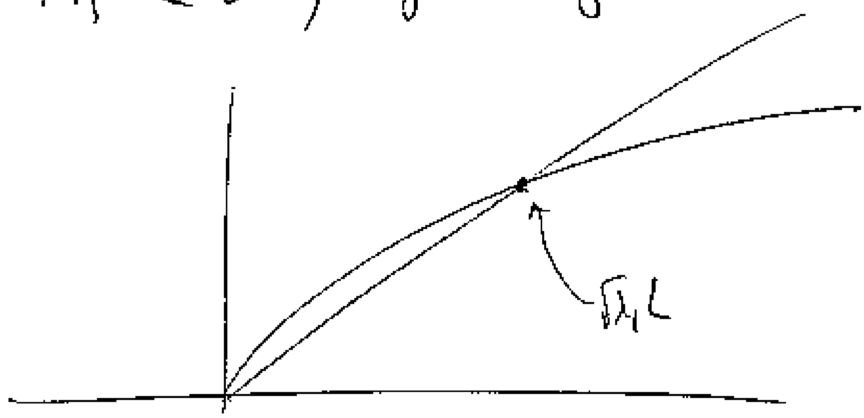
So $\pi < \sqrt{\lambda_2}L < \frac{3\pi}{2}$. Hence

$\varphi_2 = \sin\sqrt{\lambda_2}x$ has 1 zero in $(0,L)$

What about

<u>$hL < -1$</u>

$\lambda_1 < 0$, given by.



So $\varphi_1 = \sinh\sqrt{\lambda_1}x$ ← no zeros

But $\varphi_2 = \sin\sqrt{\lambda_2}x$ where $\sqrt{\lambda_2}$ is given by