Math 361 2006

In 260 we learned about ordinary differential eqns.

Eg $$\frac{dX}{dt} = -\alpha X \quad , \quad X(0) = X_0 .$$

This has solution $$X(t) = X_0 e^{-\alpha t}$$

note: solution is function of time.

Could apply to, for instance, radioactive decay, cooling etc. //

But many things depend on space and time. Example: just about anything you like.

Spse u depends on space and time. A P.D.E. is a diff. eq. that relates the partial derivatives of u.

Eg $$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$ (Diffusion, or Heat, Eqn)

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a)$$ FitzHugh Nagumo. (Reaction - Diffusion)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace eqn)}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \text{(Wave eqn)}$$

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Study of PDEs is a lot more difficult than ODEs. We study a number of important simple linear cases in this course.

Derivation of Heat Eqn (Often called the Diffusion Eqn) $\boxed{\text{(Section 1.2)}}$

Arbitrary region 1-d Ω. $u(x, t)$ = heat at point $x \in \Omega$ at time t.

Total heat is $\int_\Omega u(x,t)\,dx$

(integral over the length)

Conservation law: rate of change total = flux over boundary

flux of heat in is $-D \frac{\partial u}{\partial x}(x,t)$

flux of heat ~~out~~ in is $D \frac{\partial u}{\partial x}(x+\Delta x, t)$

x $\qquad$ $x+\Delta x$

Newton's law of cooling. Or Fick's Law. From empirical observations

Now apply conservation

$$\frac{d}{dt}\int_{x}^{x+\Delta x} u(s,t)\,ds = D\frac{\partial u}{\partial x}(x+\Delta x,t) - D\frac{\partial u}{\partial x}(x,t)$$

$$\int_{x}^{x+\Delta x} \frac{\partial u}{\partial t}(s,t)\,ds = D\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right)$$

M.V.T. $\quad \Delta x\,\dfrac{\partial u}{\partial t}(\xi,t) = D\left(\qquad\qquad\right), \quad \xi \in [x, x+\Delta x]$

Let $\Delta x \to 0$ $\quad \dfrac{\partial u}{\partial t}(x,t) = D\dfrac{\partial^2 u}{\partial x^2}$ (assuming D = constant)

P.D.E.s need initial and boundary conditions.

Eg

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 1 \end{aligned}\right\} \text{ ends held at fixed temp.}$$

$u(x,0) = f(x)$ (initial heat distribution).

Note: ODEs need only initial conditions.

Question

How do we solve such eqns? Need completely different methods than for ODEs. First, a crucial concept.

Linearity $\boxed{\text{Section 2.2}}$

Linear operator L is such that $L(c_1 u_1 + u_2) = c_1 L(u_1) + L(u_2)$

Eg – $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$ are all linear operators.

– $L(u) = u^2$ is not linear.

– $L(u) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}$ is a linear operator. (Heat or diffusion operator)

A linear equation is $L(u) = f$, for L some lin. op.

f is some known function of x and t

ie. $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, $f = \alpha(x,t)\, u$

$$\Rightarrow \quad L(u) = f \quad \Longleftrightarrow \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha(x,t)\, u. \quad \text{Linear PDE.}$$

But $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0$ is non linear.

If $f = 0$

$L(u) = 0$ is a homogeneous PDE.

From now on we always suppose that L denotes a linear operator.

Note: $L(u)=0$ always has a solution $u=0$

Pf: $L(0) = L(u-u) = L(u) - L(u) = 0$

Eg: Give examples.

Superposition If u_1 + u_2 satisfy $L(u)=0$, then so does

$$c_1 u_1 + c_2 u_2 .$$

i.e. we can add solutions of lin. hom. systems to get other solutions.

Note: this doesn't happen for nonlinear eqns.

We are going to use this many times in this course.

Separation of Variables $\boxed{\text{Section 2.3.2}}$

Eg. $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0$$

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \text{ boundary conditions}$$

$u(x,0) = f(x)$ initial condition

Solution is clear from physical grounds. $u \to 0$ everywhere, as $t \to \infty$.

Trick: look for solutions of form

$$u(x,t) = \varphi(x)\, T(t)$$

$$\Downarrow$$

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = \lambda$$

λ must be a constant.

When can we solve these eqns?

$$\frac{d^2\varphi}{dx^2} = (\lambda)\varphi.$$

$\boxed{\text{Section 2.3.4}}$

B.Cs are $\varphi(0) = 0$, $\varphi(L) = 0$ (Explain)

This has solutions only for some λ.

Try $\lambda > 0$, say $\lambda = +\mu^2$.

Solve $\dfrac{d^2\varphi}{dx^2} = \mu^2\varphi$. i.e. $\varphi = Ae^{\mu x} + Be^{-\mu x}$, for constants A, B

A + B from B.C.

$$\left.\begin{aligned} \varphi(0) = 0 &\Rightarrow A + B = 0 \\ \varphi(L) = 0 &\Rightarrow Ae^{\mu L} + Be^{-\mu L} = 0 \end{aligned}\right\} \Rightarrow A = B = 0$$

So $\lambda = \mu^2$ gives only trivial solutions which we reject.

What about $\lambda < 0$, $\lambda = -\mu^2$ say.

We need to solve

$$\frac{d^2\varphi}{dx^2} = -\mu^2\varphi \quad , \quad \varphi(0) = \varphi(L) = 0$$

Soln is $\varphi = A\sin\mu x + B\cos\mu x$.

$$\varphi(0) = 0 \Rightarrow B = 0 \Rightarrow \varphi = A\sin\mu x$$

$$\varphi(L) = 0 \Rightarrow A\sin\mu L = 0$$

$$\Rightarrow \mu L = 0, \pm\pi, \pm 2\pi \text{ etc.}$$

$$\mu = \pm\frac{n\pi}{L}. \longleftarrow \text{eigenvalues.}$$

<u>Crucial fact</u>: Solving the BVP gives solutions <u>only</u> for certain values of the separation constant λ.

For any n $\quad \sin\left(\frac{n\pi x}{L}\right)$ is a solution for φ
Called <u>eigenfunctions</u>

<u>Solving for T</u> Now we know what λ is $\left(\lambda = -\frac{n^2\pi^2}{L^2}\right)$, we can solve for T.

$$\frac{dT}{dt} = k\lambda T = -\frac{kn^2\pi^2}{L^2}T$$

$$\Rightarrow T = e^{-\frac{kn^2\pi^2}{L^2}t}$$

Now put them together

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2}{L^2}t} \qquad \text{for any } n = 0, \pm 1, \pm 2.$$

Thus, we get an <u>infinite</u> number of possible solutions, called $u_n(x,t)$.

<u>Initial Condition</u>. One thing we haven't yet done is satisfy

$$u(x,0) = f(x).$$

For any one choice of n, $u_n(x,t)$ doesn't work.

<u>But</u> (crucial thing) we can add the u_ns and still get a solution.

i.e. $u_0 + u_1$ is a solution still

$u_0 + u_1 + u_2$ is also.

etc.

<u>Question</u>. Can we add up all the u_ns in a clever way so that the sum satisfies the initial condition?

So, let

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{B_n}_{\text{some constants}} u_n(x,t) = \underbrace{\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2}{L^2}t}}_{\text{this is still a solution}}$$

At $t=0$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \stackrel{?}{=} f(x)$$

Can we choose the

Sections 2.3.5 & 2.3.6

Easy for some initial conditions. If, say,

$$f(x) = 4\sin\frac{3\pi x}{L} \quad \Rightarrow \quad B_3 = 4 \quad \text{all other } B_n = 0.$$

Next fact: Just about any function $f(x)$ can be expanded as a series of sin functions (or cosines, or both). Hence we can find the B_ns.

Trick: orthogonality.

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$$

Thus

$$f(x)\sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

Then integrate, to get

$$\frac{L}{2}B_m = \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \quad \text{\& so } B_m = \dots$$

Hence

$$f(x) = \sum_{m=1}^{\infty} \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \;\sin\left(\frac{m\pi x}{L}\right)$$

& so

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\left[\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{kn^2\pi^2}{L^2}t}$$

Example

Section 2.3.7

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \quad \text{B.C.}$$

$$u(x,0) = 100 \qquad \text{I.C.}$$

As before, $$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2}{L^2}t}$$

where $$B_n = \frac{2}{L}\int_0^L 100 \sin\frac{n\pi x}{L}\,dx = \frac{200}{n\pi}\left(1-\cos n\pi\right)$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{400}{n\pi}, & n \text{ odd} \end{cases}$$

Summary of method

- Separate variables & get different ODEs for t and x
- Use boundary conditions to solve the ODE in x
  (This will give values for the separation constant)
- Calculate the eigenfunctions & eigenvalues
- Solve the t part
- Write solution down as infinite sum.
- Finally, handle the initial condition by expanding in eigenfunctions.

But many questions.

- ~~Why~~ Why only discrete e-values?
- Why can you expand $f(x)$ in that infinite sum?
- Will that orthogonality trick always work?
- Convergence of sum?
- Discontinuous $f(x)$? What then?

These questions motivate the study of Fourier Series (Fourier, c1810)

We are going to show

- Yes, you can expand $f(x)$ using sines & cosines & this sum converges in a precise way
- Eigenfunctions arising from many PDEs work in the same way as sines & cosines. They are just the canonical example.