Maths 361

Today's topic:
Complex Fourier series
General Fourier series

Recommended reading:
Haberman §3.6

Section 1.6 Complex Fourier series

Our results about Fourier representations can be extended to complex valued functions of a real variable $f : \mathbf{R} \to \mathbf{C}$. Write $f(x) = u(x) + iv(x)$ where $i = \sqrt{-1}$ and u and v are real valued functions of the real variable x.

We can integrate and differentiate complex valued functions as usual.

Let $\bar{f}$ denote the complex conjugate of f, i.e., for $f(x) = u(x) + iv(x)$, $\overline{f}(x) = u(x) - iv(x)$. We now modify our default IPS to become $PS[-L, L]$ with inner product defined by

$$\langle f, g \rangle = \int_{-L}^{L} \overline{f}(x) g(x) \, dx$$

This IP satisfies the axioms

1. $\langle f, g \rangle = \overline{\langle g, f \rangle}$
2. $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
3. $\langle cf, g \rangle = c \langle f, g \rangle$
4. $\langle f, f \rangle \geq 0$ with $\langle f, f \rangle = 0$ if and only if $f = 0$.

Claim : The set

$$\left\{ e^{-in\pi x/L} \right\}_{n=-\infty}^{\infty}$$

is an orthogonal set for the default IPS.

This orthogonal set can be used to formally write down a Fourier representation of a function f :

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \, e^{-in\pi x/L}$$

where c_n is a complex constant for each integer n. Note that if f is real-valued, this representation is just another way of writing the real trig Fourier representation:

The set

$$\left\{e^{-in\pi x/L}\right\}_{n=-\infty}^{\infty}$$

is complete for $PS[-L, L]$. We can calculate the coefficients c_n directly:

So

$$\begin{aligned} c_m &= \frac{\langle e^{-im\pi x/L}, f\rangle}{\|e^{-im\pi x/L}\|^2} \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{im\pi x/L}\,dx \end{aligned}$$

Example : Calculate the complex trig Fourier representation of

$$f(x) = \begin{cases} 0, & -L \le x < 0, \\ x, & 0 \le x < L. \end{cases}$$

Hence, the complex Fourier representation of f is

$$\frac{L}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{L}{2n\pi}\left[\frac{(-1)^n - 1}{n\pi} - i(-1)^n\right] e^{-in\pi x/L}$$

Maple can be used to plot a sum of terms in the representation.