Maths 361

Today's topics:
Approximation by Fourier series
L^2 convergence of Fourier series

Questions motivating today's lecture

(1) We frequently approximate a function by a partial sum of its Fourier series. How good is such an approximation?

(2) For a function $f \in PS[-L, L]$ its real trig Fourier series converges pointwise to f at points of continuity of f and behaves nicely at discontinuities. What can we say in the case $f \notin PS[-L, L]$?

Section 1.6 Approximation by Fourier series

If V is an inner product space with inner product $\langle f, g\rangle$ and norm $\|f\|$ then the quantity $\|f - g\|$ is a measurement of the "distance" between two functions f and g.

Example: For the IPS $PS[a, b]$ with inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

and induced norm

$$\|f\| = \sqrt{\int_a^b (f(x))^2\,dx}$$

this "distance" is

$$\|f - g\| = \sqrt{\int_a^b (f(x) - g(x))^2\,dx}$$

(This is sometimes called the L^2 *metric*.)

Theorem (Least-Squares Approximation)

Let $\{\phi_n\}_{n=1}^{\infty}$ be any complete orthogonal set of functions for the IPS V. Consider $f \in V$ with $\|f\| < \infty$. Let N be a *fixed* positive integer. Write

$$S_N(x) = \sum_{n=1}^{N} d_n \phi_n(x)$$

for arbitrary constants $d_n,\ n = 1, 2, \ldots N$, and let

$$c_n^f = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2}.$$

The choice $d_n = c_n^f$ minimises

$$\|S_N - f\|$$

over all possible choices of the constants d_n.
To see this, let $E = \|S_N - f\|$.

$$\begin{aligned}E^2 &= \|S_N - f\|^2 = \langle S_N - f, S_N - f\rangle \\ &= \langle S_N, S_N - f\rangle - \langle f, S_N - f\rangle \\ &= \langle S_N, S_N -\rangle - \langle S_N, f\rangle - \langle f, S_N\rangle + \langle f, f\rangle \\ &= \|S_N\|^2 - 2\langle S_N, f\rangle + \|f\|^2\end{aligned}$$

But

$$\begin{aligned}\|S_N\|^2 &= \langle \sum_{n=1}^{N} d_n\phi_n, \sum_{m=1}^{N} d_m\phi_m\rangle \\ &= \sum_{n=1}^{N} d_n\langle\phi_n, \sum_{m=1}^{N} d_m\phi_m\rangle \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N} d_n d_m\langle\phi_n, \phi_m\rangle \\ &= \sum_{n=1}^{N} d_n^2\langle\phi_n, \phi_n\rangle \quad \text{because } \langle\phi_n, \phi_m\rangle = 0 \text{ if } n \neq m \\ &= \sum_{n=1}^{N} d_n^2\|\phi_n\|^2.\end{aligned}$$

Also

$$\langle S_N, f\rangle = \langle\sum_{n=1}^{N} d_n\phi_n, f\rangle = \sum_{n=1}^{N} d_n\langle\phi_n, f\rangle.$$

So we need to find the minimum of

$$E^2 = \|f\|^2 + \sum_{n=1}^{N}\left(d_n^2\|\phi_n\|^2 - 2d_n\langle\phi_n, f\rangle\right)$$

Now complete the square for the nth term of this sum:

$$\begin{aligned}d_n^2\|\phi_n\|^2 - 2d_n\langle\phi_n, f\rangle &= \|\phi_n\|^2\left(d_n^2 - 2\frac{\langle\phi_n, f\rangle}{\|\phi_n\|^2}d_n\right) \\ &= \|\phi_n\|^2\left(d_n - \frac{\langle\phi_n, f\rangle}{\|\phi_n\|^2}\right)^2 - \frac{\langle\phi_n, f\rangle^2}{\|\phi_n\|^2}\end{aligned}$$

This quadratic expression is clearly minimised when

$$d_n = \frac{\langle\phi_n, f\rangle}{\|\phi_n\|^2} = c_n^f$$

The minimum value of E^2 is

$$E_{\min}^2 = \|f\|^2 - \sum_{n=1}^{N}\frac{\langle\phi_n, f\rangle^2}{\|\phi_n\|^2}$$

From here on, let

$$S_N^f(x) = \sum_{n=1}^{N} c_n^f \phi_n(x)$$

where

$$c_n^f = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}$$

and $\{\phi_n\}_{n=1}^{\infty}$ is an orthogonal set of functions in an IPS V.

Claim :

$$\|S_N^f\|^2 \leq \|f\|^2$$

This is called **Bessel's Inequality**.

To see this, use the orthogonality:

$$\begin{aligned}
\|S_N^f\|^2 &= \langle \sum_{n=1}^{N} c_n^f \phi_n, \sum_{m=1}^{N} c_m^f \phi_m \rangle \\
&= \sum_{n=1}^{N} (c_n^f)^2 \|\phi_n\|^2 \\
&= \sum_{n=1}^{N} \frac{\langle \phi_n, f \rangle^2}{\|\phi_n\|^2} \\
&= \|f\|^2 - E_{\min}^2 \quad \text{from the previous page} \\
&\leq \|f\|^2.
\end{aligned}$$

Definition : The Fourier representation of f, S_N^f, **converges in the norm** to f if

$$\lim_{N \to \infty} \|S_N^f - f\| = 0.$$

Definition : Let V be an IPS. The set $\{\phi_n\}_{n=1}^{\infty}$ is **complete for convergence in the norm** in V if each $f \in V$ has a Fourier representation S_N^f converging to f in the norm.

Example : Our default norm

$$\|f\| = \sqrt{\int_a^b (f(x))^2 \, dx}$$

is called the L^2 or "mean square" norm. So our default IPS, $PS[a, b]$ with

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

is called "$PS[a, b]$ with L^2 norm". Another important IPS of functions is

$$L2[a, b] = \{ f : [a, b] \to \mathbf{R} \mid \ \|f\| < \infty \}$$

where $\|f\|$ is the L^2 norm of f.

Theorem (L2 convergence of Fourier series)

The set
$$\left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$$
is complete for convergence in the mean square norm for $L2[-L, L]$.

Summary of convergence results:
(a) If $f \in PS[-L, L]$ then the real trig Fourier series of f converges pointwise to f at almost all $x \in [-L, L]$ and behaves nicely at points of discontinuity, and also converges to f in the mean square sense.
(b) If $f \in L2[-L, L]$ then the real trig Fourier series of f converges to f in the mean square sense and converges pointwise to f at points of continuity of f. We do not have a result about the behaviour at discontinuities of f.

A consequence of the previous theorem is **Parseval's Theorem**

If $f \in L2[-L, L]$ and S_∞^f is the real trig Fourier series of f, then
$$\|S_\infty^f\| = \|f\|.$$

Recall from the proof of Bessel's inequality that
$$\|S_N^f\|^2 = \|f\|^2 - E_{\min}^2 = \|f\|^2 - \|S_N - f\|^2$$

But $\|S_N - f\|^2 \to 0$ as $N \to \infty$.