Maths 361 Fourier Series Notes 2

Today's topics:
Even and odd functions
Real trigonometric Fourier series

Section 1.2 : Odd and even functions
Consider a function $f : [-L, L] \to \mathbf{R}$.

- f is **odd** if $f(-x) = -f(x)$ for all $x \in [-L, L]$.
- f is **even** if $f(-x) = f(x)$ for all $x \in [-L, L]$.

Some useful properties of odd and even functions :

1. If f is even
$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$
If f is odd
$$\int_{-L}^{L} f(x)\,dx = 0$$
2. If f, g are even functions and q, r are odd functions then fg and qr are even functions, fq is an odd function.

A function defined on the interval $[0, L]$ can be extended to $[-L, L]$ as an even function or an odd function:
The **odd extension of** f is defined by

$$f_{\text{odd}}(x) = \begin{cases} f(x), & x \in [0, L], \\ -f(-x), & x \in [-L, 0) \end{cases}$$

The **even extension of** f is defined by

$$f_{\text{even}}(x) = \begin{cases} f(x), & x \in [0, L], \\ f(-x), & x \in [-L, 0) \end{cases}$$

Section 1.3 : Real trig Fourier series

Recall from last lecture :

If $\{\phi_n\}_{n=1}^{\infty}$ is a complete orthogonal set in $PS[a,b]$ then $f \in PS[a,b]$ can be represented as a sum of ϕ_n:

$$f(x) = c_n\phi_n = \sum_{n=1}^{\infty} \frac{\langle f(x), \phi_n(x)\rangle}{\|\phi_n\|^2}\phi_n(x)$$

Equality will hold except possibly for a finite number of points $x \in [a,b]$.

We would like to express a function $f(x)$ as a sum of cosine and sine terms, i.e., would like

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

for some a_0, a_n, b_n, $n = 1, 2\ldots$

The right side of equation (1) is the called the **real trig Fourier representation** of $f(x)$.

Assuming (for now) no problems with convergence of the infinite series, we can calculate a_0, a_n, b_n using earlier results. In particular,

$$S = \left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$$

is an orthogonal set in the IPS $PS[-L,L]$ with inner product $\langle f, g, \rangle = \int_{-L}^{L} f(x)g(x)\,dx$, and is in fact complete (for pointwise convergence). So if $f \in PS[-L,L]$ then

$$a_0 = \frac{\langle f, 1\rangle}{\|1\|^2}$$

$$a_n = \frac{\langle f, \cos\left(\frac{n\pi x}{L}\right)\rangle}{\|\cos\left(\frac{n\pi x}{L}\right)\|^2}$$

$$b_n = \frac{\langle f, \sin\left(\frac{n\pi x}{L}\right)\rangle}{\|\sin\left(\frac{n\pi x}{L}\right)\|^2}$$

for $n = 1, 2\ldots$.

We can calculate $\|1\|^2 = 2L$, $\|\cos\left(\frac{n\pi x}{L}\right)\|^2 = L$ and $\|\sin\left(\frac{n\pi x}{L}\right)\|^2 = L$ for $n = 1, 2, \ldots$.

Summary : A function $f \in PS[-L,L]$ has the (classical or real trig) Fourier representation

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

for $n = 1, 2 \ldots$

Note that $\sim$ in (2) means 'has the Fourier representation'. We do not (yet) know whether f is equal to its Fourier representation at any particular point x although we expect this to be the case at all but finitely many points of $[-L, L]$.

Example : Calculate the real trig Fourier representation of

$$f(x) = 1 + x \text{ for } x \in [-L, L]$$

We find that

$$f(x) \sim 1 + \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right)$$

i.e., $a_0 = 1$, all other $a_n = 0$, $b_n = \frac{2L}{n\pi}(-1)^{n+1}$

We can use Matlab to plot the first few terms of the Fourier series:

Example : Calculate the real trig Fourier representation of

$$f(x) = \begin{cases} -1, & -L \leq x < 0 \\ 1, & 0 \leq x \leq L \end{cases}$$

We find that

$$f(x) \sim \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

i.e., all $a_n = 0$, $b_n = \frac{4}{n\pi}$ for odd n and $b_n = 0$ for even n.

We can use Matlab to plot the first few terms of the Fourier series:

Today's topics:
Convergence of Fourier series
Sketching Fourier series
Fourier cosine and sine series

Recommended reading:
Haberman §3.2 and §3.3 (excluding Gibb's phenomenon)

Recommended exercises:
Haberman 3.2.1(b),(d),(f); 3.2.2(a),(c),(f); 3.3.1(d); 3.3.2(d); 3.3.4; 3.3.5

Section 1.4 Convergence of Fourier series

Consider a function $f : [a, b] \to \mathbf{R}$. We define f_{per}, the **periodic extension of** f, by

$$f_{\text{per}}(x + n(b - a)) = f(x)$$

for each integer n and each $x \in [a, b)$.

Theorem (Dirichlet) : Let $f \in PS[-L, L]$. Let

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

for $n = 1, 2 \ldots$.

Define

$$S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right),$$

i.e., S_N is the sum of the first $2N + 1$ terms in the real trig Fourier representation of f for $x \in [-L, L]$.

Then

$$\lim_{N\to\infty} S_N(x) = \frac{f_{\text{per}}(x^+) + f_{\text{per}}(x^-)}{2}$$

for $x \in \mathbf{R}$.

Dirichlet's theorem says that for $f \in PS[-L, L]$, the Fourier series of f converges to

- f_{per} whenever f_{per} is continuous;
- the average of the right and left limits of f_{per} whenever f_{per} has a jump discontinuity.

See section 5.5 of "PDE's : an introduction" by W. A. Strauss for a proof of this theorem.

Sketching Fourier series We can now sketch the Fourier representation of a function without first calculating a_0, a_n, b_n.

For $f \in PS[a, b]$, let

$$S_\infty(x) = \lim_{N\to\infty} S_N$$

be the real trig Fourier representation of f in $[-L, L]$. To sketch $S_\infty(x)$:

1. Sketch $f_{\text{per}}(x)$ without marking value at points of discontinuity.
2. Mark
$$\frac{f_{\text{per}}(x^+) + f_{\text{per}}(x^-)}{2}$$
at points of discontinuity.

§1.5 Fourier sine and cosine series

The sets

$$S_1 = \left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$$

and

$$S_2 = \left\{ 1, \cos\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$$

are orthogonal and complete for pointwise convergence in $PS[0,L]$ with inner product

$$\langle f, g \rangle = \int_0^L f(x)g(x)\,dx$$

Define the **Fourier sine series of** f to be

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Define the **Fourier cosine series of** f to be

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

Theorem : If $f \in PS[0,L]$, then at each $x \in \mathbf{R}$ the Fourier sine series of f converges to

$$\frac{f_{\text{per}}(x^+) + f_{\text{per}}(x^-)}{2}$$

where f_{per} is the periodic extension of f_{odd} and f_{odd} is the odd extension of f to $[-L, L]$ as defined in the last lecture.

Similarly, if $f \in PS[0,L]$, then at each $x \in \mathbf{R}$ the Fourier cosine series of f converges to

$$\frac{f_{\text{per}}(x^+) + f_{\text{per}}(x^-)}{2}$$

where f_{per} is the periodic extension of f_{even} and f_{even} is the even extension of f to $[-L, L]$ as defined in the last lecture.