Preliminaries

Notation

By now you should be familiar with notation such as "$\subseteq$" and "$=$". However, below we present some notation that may not be so familiar to you.

$\in$ — is a member of

$\leqslant$ — less than or equal to

$\cup$ — union (of sets)

$\cap$ — intersection (of sets)

$:=$ — equals, by definition

$\mathbb{N}$ — the set of all natural numbers $1, 2, 3, \ldots$

$\mathbb{Z}$ — the set of all integers $0, \pm 1, \pm 2, \pm 3, \ldots$

$\mathbb{Q}$ — the set of all rational numbers

$\mathbb{R}$ — the set of all real numbers consisting of both $\mathbb{Q}$ and the irrationals

On the real number line there are several different types of intervals:

open — $(a, b) := \{x \in \mathbb{R} : a < x < b\}$

closed — $[a, b] := \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$

half open on left — $(a, b] := \{x \in \mathbb{R} : a < x \leqslant b\}$

half open on right — $[a, b) := \{x \in \mathbb{R} : a \leqslant x < b\}$

infinite intervals — $(a, \infty) := \{x \in \mathbb{R} : a < x\}$

— $[a, \infty) := \{x \in \mathbb{R} : a \leqslant x\}$

— $(-\infty, b) := \{x \in \mathbb{R} : x < b\}$

— $(-\infty, b] := \{x \in \mathbb{R} : x \leqslant b\}$

— $(-\infty, \infty) := \mathbb{R}$

Note that although $\mathbb{R}$ may be represented as $(-\infty, \infty)$ it should be stressed that $-\infty$ and ∞ are **not** "real" numbers and should not be treated as such. Thus expressions such as $\infty - \infty$ and ∞/∞ have no meaning.

Inequalities

The majority of the work in this course requires the manipulation of inequalities. So below we record some of the basic properties possessed by the binary relation " $\leqslant$ ".

Theorem:

(i) *for every* a, $b \in \mathbb{R}$ *either,* $a \leqslant b$ *or* $b \leqslant a$*;*

(ii) *if* $a \leqslant b$ *and* $b \leqslant a$ *then* $a = b$*;*

(iii) *if* $a \leqslant b$ *and* $b \leqslant c$ *then* $a \leqslant c$*;*

(iv) *if* $a \leqslant b$ *and* $c \in \mathbb{R}$ *then* $a + c \leqslant b + c$*;*

(v) *if* $a \leqslant b$ *and* $0 \leqslant c$ *then* $ac \leqslant bc$*.*

Exercise: Show that for every $a \in \mathbb{R}$, $0 \leqslant a^2$.

Absolute values

The *absolute value* of a number $a \in \mathbb{R}$ is denoted by $|a|$. It is formally defined by,

$$|a| := \begin{cases} a & 0 \leqslant a; \\ -a & a < 0. \end{cases}$$

Hence $0 \leqslant |a|$. Some properties of the absolute value are given in the following theorem.

Theorem: *Suppose* $a, b \in \mathbb{R}$*. Then;*

(i) $-|a| \leqslant a \leqslant |a|$ *and* $a = |a|$ *if, and only if,* $0 \leqslant a$*;*

(ii) $|ab| = |a||b|$*;*

(iii) $|a + b| \leqslant |a| + |b|$ *(the triangle inequality);*

(iv) $\Big||a| - |b|\Big| \leqslant |a - b|$ *(the reverse triangle inequality).*

Proof: The proofs of (i) and (ii) are left as exercises. (iii) Let a and b be real numbers, then $|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 \leqslant a^2 + |2ab| + b^2 = |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2$. The result now follows by taking the square root of this equation. The proof of (iv) is left as an exercise. $\square$

Least upper bound principle

Let $\mathcal{S}$ be a non-empty subset of $\mathbb{R}$. Then we say that $\mathcal{S}$ is *bounded above* if there exists an $M \in \mathbb{R}$ such that $s \leqslant M$ for all $s \in \mathcal{S}$.

For a non-empty subset $\mathcal{S}$ of $\mathbb{R}$ any real number $M \in \mathbb{R}$ such that $s \leqslant M$ for all $s \in \mathcal{S}$ is called an *upper bound* for $\mathcal{S}$.

Let $\mathcal{S}$ be a non-empty subset of $\mathbb{R}$. Then we say that $\mathcal{S}$ is *bounded below* if there exists an $m \in \mathbb{R}$ such that $m \leqslant s$ for all $s \in \mathcal{S}$.

For a non-empty subset $\mathcal{S}$ of $\mathbb{R}$ any real number $m \in \mathbb{R}$ such that $m \leqslant s$ for all $s \in \mathcal{S}$ is called a *lower bound* for $\mathcal{S}$.

Let $\mathcal{S}$ be a non-empty subset of $\mathbb{R}$. Then we say that $\mathcal{S}$ is *bounded* if it is bounded above and bounded below.

Exercise: Show that a non-empty subset $\mathcal{S}$ of $\mathbb{R}$ is bounded if, and only if, there exists an $M \in \mathbb{R}$ such that $|s| \leqslant M$ for all $s \in \mathcal{S}$.

Least upper bound principle: Let $\mathcal{S}$ be a non-empty subset of $\mathbb{R}$ that is bounded above. Then $\mathcal{S}$ has a *least upper bound* (sometimes called the *supremum*), denoted $\sup \mathcal{S}$.

"This means: that for every non-empty subset $\mathcal{S}$ of $\mathbb{R}$ that is bounded above there exists an upper bound M such that if M' is any other upper bound for $\mathcal{S}$ then $M < M'$, *ie:* M is the smallest of all the upper bounds for $\mathcal{S}$."

Exercise: Show that every non-empty set of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (sometimes called the *infimum*), denoted $\inf \mathcal{S}$. That is, show that for every non-empty subset $\mathcal{S}$ of $\mathbb{R}$ that is bounded below there exists a lower bound m such that if m' is any other lower bound for $\mathcal{S}$ then $m' < m$, *ie:* m is the largest of all the lower bounds for $\mathcal{S}$.

Exercise: Let P denote the set of all positive real numbers, ie: $P := \{a \in \mathbb{R} : 0 < a\}$. Show that 0 is the greatest lower bound for P.

Proposition: Let a and b be any two real numbers. Then $a \leqslant b$ if, and only if, for each $0 < \varepsilon$, $a \leqslant b + \varepsilon$.

Proof: Clearly if $a \leqslant b$ then for each $0 < \varepsilon$, $a \leqslant b + \varepsilon$. So we need only consider the converse question. To this end let us suppose that for each $0 < \varepsilon$, $a \leqslant b + \varepsilon$. Then for each $0 < \varepsilon$, $a - b \leqslant \varepsilon$ and so $a - b$ is a lower bound for the positive real numbers, but by the above exercise this means that $a - b \leqslant 0$ since 0 is the greatest lower bound for the positive real numbers. Hence $a \leqslant b$. $\square$

The "triangle inequality" and "least upper bound principle" are the corner stones of "Real analysis".

Sequences

Introduction

A *sequence* (of real numbers) is any function from $\mathbb{N}$ into $\mathbb{R}$. Historically, people have used the notation $(a_n : n \in \mathbb{N})$ instead of writing $a : \mathbb{N} \to \mathbb{R}$ and it is standard to write a_n in place of $a(n)$. For any such sequence we shall define, for each $N \in \mathbb{N}$, the *Nth-tail* of $(a_n : n \in \mathbb{N})$ to be the sequence $(t_n : n \in \mathbb{N})$ given by, $t_n := a_{N+n}$ and we shall say that a sequence $(t_n : n \in \mathbb{N})$ is a *tail* of a sequence $(a_n : n \in \mathbb{N})$ if $(t_n : n \in \mathbb{N})$ is the N-th tail of $(a_n : n \in \mathbb{N})$ for some $N \in \mathbb{N}$.

If P is a property that a sequence may or may not have then we may say that a sequence $(a_n : n \in \mathbb{N})$ *eventually* has property P if for some $N \in \mathbb{N}$ the N-th tail of $(a_n : n \in \mathbb{N})$ has property P, ie: if some tail of $(a_n : n \in \mathbb{N})$ has property P.

A real number L is a called a *limit* of a sequence $(a_n : n \in \mathbb{N})$ if for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n > N$. If L is a limit of a sequence $(a_n : n \in \mathbb{N})$ then we write: $\lim\limits_{n\to\infty} a_n = L$.

We say that a sequence $(a_n : n \in \mathbb{N})$ is *convergent* if there exists an $L \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ so that $|a_n - L| < \varepsilon$ for all $n > N$.

Intuitively, a real number L is the limit of a sequence $(a_n : n \in \mathbb{N})$ if given any degree of precision (usually denoted by ε) there exists a point (usually denoted by N) such that once we are past it (ie: $n > N$) a_n approximates L to within this prescribed degree of accuracy, ie: a_n is within ε of L, or $|a_n - L| < \varepsilon$. Alternatively, we could simply say that L is the limit of $(a_n : n \in \mathbb{N})$ if for each $\varepsilon > 0$, $(a_n : n \in \mathbb{N})$ is "eventually" within ε of L.

Exercise: Show that a sequence $(a_n : n \in \mathbb{N})$ converges to a real number L if, and only if, for some $N \in \mathbb{N}$ the N-th tail of $(a_n : n \in \mathbb{N})$ converges to L, ie: if, and only if, the "tail" of $(a_n : n \in \mathbb{N})$ converges to L.

Example: Use a formal ε - N argument to show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := 1/n$ is convergent.

Answer: Let ε be any positive real number. If N is defined to be any natural number greater than $1/\varepsilon$ then,

$$|a_n - 0| = |1/n - 0| = |1/n| = 1/n < 1/N < \varepsilon$$

for all $n > N$. Thus we have shown that the sequence $(a_n : n \in \mathbb{N})$ converges to 0. $\square$

Example: Use a formal ε - N argument to show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := (-1)^n$ is **not** convergent.

Answer: We shall use a proof by contradiction. So let us assume, in order to obtain a contradiction, that the sequence $(a_n : n \in \mathbb{N})$ converges to some real number L. Then by the definition of a limit there exists an $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for all $n > N$, ie: we have substituted $\varepsilon = 1$ into the definition of a limit. Now,

$$|a_{n+1} - a_n| \leqslant |a_{n+1} - L| + |a_n - L| < 1 + 1 = 2 \quad \text{for all } n > N.$$

However, $|a_{n+1} - a_n| = |(-1)^{n+1} - (-1)^n| = |(-1)^{n+1} + (-1)^{n+1}| = 2|(-1)^{n+1}| = 2$ for all $n \in \mathbb{N}$. Hence we have obtained a contradiction and so the sequence $(a_n : n \in \mathbb{N})$ is **not** convergent. $\square$

Example: Use a formal ε - N argument to show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := \sqrt{n^2+2} - n$ is convergent.

Answer: We shall prove that $\lim\limits_{n\to\infty} a_n = 0$. To this end, let ε be any positive real number and let N be any natural number greater than $2/\varepsilon$. Then,

$$\begin{aligned} |a_n - 0| &= |\sqrt{n^2+2} - n| \\ &= \left| \frac{(\sqrt{n^2+2} - n)(\sqrt{n^2+2} + n)}{\sqrt{n^2+2} + n} \right| \\ &= \frac{2}{\sqrt{n^2+2} + n} < \frac{2}{n} < \frac{2}{N} < \varepsilon \quad \text{for all } n > N. \qquad \square \end{aligned}$$

Observation: Let $(a_n : n \in \mathbb{N})$ be a sequence and let L be a real number. If it is possible to show that "$|a_n - L| \leqslant K/n^p$" for some positive real number K and natural number p, for all $n \in \mathbb{N}$, then $(a_n : n \in \mathbb{N})$ converges to L.

Proof: Let ε be any positive real number. If we set N to be any natural number greater than $\sqrt[p]{K/\varepsilon}$ then, $|a_n - L| \leqslant K/n^p < K/N^p < \varepsilon$ for all $n > N$. $\square$

Exercise: Prove, using an ε - N argument that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := 1 + \dfrac{(-1)^n}{n^2}$ converges to 1.

Theorem: *If $(a_n : n \in \mathbb{N})$ is a convergent sequence then it has only one limit, ie: its limit is unique.*

Proof: Suppose that L_1 and L_2 are limits of the sequence $(a_n : n \in \mathbb{N})$. We need to show that $L_1 = L_2$, ie: $|L_1 - L_2| = 0$. Let ε be any positive real number. Then by the fact that L_1 is a limit of $(a_n : n \in \mathbb{N})$ there exists an $N_1 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon/2$ for all $n > N_1$. Similarly, by the fact that L_2 is a limit of $(a_n : n \in \mathbb{N})$ there exists an $N_2 \in \mathbb{N}$ such that $|a_n - L_2| < \varepsilon/2$ for all $n > N_2$. Hence for any $n > \max\{N_1, N_2\}$ we have that

$$|L_1 - L_2| \leqslant |a_n - L_1| + |a_n - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now since our choice of $\varepsilon > 0$ was arbitrary it must be the case that $|L_1 - L_2| = 0$. $\square$

The following theorem enables us to show that certain sequences converge without having to resort to using ε's and N's.

Theorem: *Let* $(a_n : n \in \mathbb{N})$ *and* $(b_n : n \in \mathbb{N})$ *be sequences and suppose* $\lim_{n\to\infty} a_n = L_1$ *and* $\lim_{n\to\infty} b_n = L_2$ *then;*

(i) $\lim_{n\to\infty} (a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right) = L_1 + L_2$*;*

(ii) $\lim_{n\to\infty} ca_n = c \cdot \left(\lim_{n\to\infty} a_n\right) = c \cdot L_1$*;*

(iii) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = L_1 \cdot L_2$*;*

(iv) $\lim_{n\to\infty} a_n \div b_n = \left(\lim_{n\to\infty} a_n\right) \div \left(\lim_{n\to\infty} b_n\right) = L_1 \div L_2$*, provided* $L_2 \neq 0$.

Proof: (i) Given $\varepsilon > 0$, choose N_1, $N_2 \in \mathbb{N}$ so that

$$|a_n - L_1| < \varepsilon/2 \text{ for all } n > N_1 \quad \text{and} \quad |b_n - L_2| < \varepsilon/2 \text{ for all } n > N_2$$

Then $|(a_n+b_n)-(L_1+L_2)| \leqslant |a_n-L_1|+|b_n-L_2| < \varepsilon/2+\varepsilon/2 = \varepsilon$ for all $n > \max\{N_1, N_2\}$.

(ii) Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $|a_n - L_1| < \varepsilon/(|c| + 1)$ for all $n > N$. Then, $|ca_n - cL_1| = |c||a_n - L_1| < \varepsilon \cdot |c|/(|c| + 1) < \varepsilon$ for all $n > N$.

(iii) Given $\varepsilon > 0$, choose N_1, $N_2 \in \mathbb{N}$ so that

$$|a_n - L_1| < \sqrt{\varepsilon} \quad \text{for all } n > N_1 \quad \text{and} \quad |b_n - L_2| < \sqrt{\varepsilon} \quad \text{for all } n > N_2.$$

Then $|(a_n - L_1)(b_n - L_2) - 0| = |a_n - L_1||b_n - L_2| < \varepsilon$ for all $n > \max\{N_1, N_2\}$. This proves that $\lim_{n\to\infty} (a_n - L_1)(b_n - L_2) = 0$. Combining this with (i) and (ii), we have that

$$\begin{aligned} \lim_{n\to\infty} (a_n b_n - L_1 L_2) &= \lim_{n\to\infty} \Big((a_n - L_1)(b_n - L_2) + L_1(b_n - L_2) + L_2(a_n - L_1)\Big) \\ &= 0 + L_1 \cdot \lim_{n\to\infty} (b_n - L_2) + L_2 \cdot \lim_{n\to\infty} (a_n - L_1) \\ &= 0 + L_1 \cdot 0 + L_2 \cdot 0 = 0 \end{aligned}$$

and so (iii) is obtained. Here we used that $\lim_{n\to\infty} a_n = L$ if, and only if, $\lim_{n\to\infty} (a_n - L) = 0$.

(iv) Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $|b_n - L_2| < |L_2|/2$ for all $n > N$. Then $|L_2|/2 < |L_2| - |b_n - L_2| \leqslant |b_n|$ for all $n > N$. Next choose, $N_1 > N$ such that $|b_n - L_2| < |L_2|^2\varepsilon/2$. Then we have that

$$|1/b_n - 1/L_2| = \frac{|b_n - L_2|}{|L_2||b_n|} < \frac{2|b_n - L_2|}{|L_2|^2} < \varepsilon \quad \text{for all } n > N_1.$$

This proves that $\lim_{n\to\infty} (1/b_n) = 1/L_2$. The result now follows by using (iii). □

Exercise: Give an alternative proof to part(iii) of the previous theorem by;

(a) showing that each convergent sequence $(a_n : n \in \mathbb{N})$ is bounded, ie: there exists and $M > 0$ such that $|a_n| < M$ for all $n \in \mathbb{N}$;

(b) using the inequality,

$$|a_n b_n - L_1 L_2| \leqslant |a_n b_n - a_n L_2| + |a_n L_2 - L_1 L_2| = |a_n||b_n - L_2| + |L_2||a_n - L_1|.$$

Example: Find the limit of the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := \dfrac{3n^2+2n+1}{2n^2+1} + \dfrac{3}{n}$.

Answer: $\lim\limits_{n\to\infty}\left(\dfrac{3n^2+2n+1}{2n^2+1} + \dfrac{3}{n}\right) = \lim\limits_{n\to\infty}\left(\dfrac{3+2/n+1/n^2}{2+1/n^2} + \dfrac{3}{n}\right)$. We now apply the above theorem to get $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\dfrac{3+2/n+1/n^2}{2+1/n^2}\right) + \lim\limits_{n\to\infty}\dfrac{3}{n} = \dfrac{3}{2} + 0 = 3/2$. □

Theorem: (Sandwich Theorem) *If $(a_n : n \in \mathbb{N})$, $(b_n : n \in \mathbb{N})$ and $(c_n : n \in \mathbb{N})$ are sequences with $a_n \leqslant b_n \leqslant c_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n$ then $(b_n : n \in \mathbb{N})$ is convergent and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$.*

Proof: Let $L := \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n$ and suppose $\varepsilon > 0$ is given. Choose N_1, $N_2 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n > N_1$ and $|c_n - L| < \varepsilon$ for all $n > N_2$. Then,

$$-\varepsilon < a_n - L \leqslant b_n - L \leqslant c_n - L < \varepsilon \quad \text{for all } n > \max\{N_1, N_2\}$$

and so $|b_n - L| < \varepsilon$ for all $n > \max\{N_1, N_2\}$. This completes the proof. □

Example: Calculate the limit of the sequence $(b_n : n \in \mathbb{N})$ defined by, $b_n := \dfrac{(-1)^n \sin(n^2)}{n}$.

Answer: First note that $|b_n| \leqslant 1/n$ for all $n \in \mathbb{N}$. Then define the sequences $(a_n : n \in \mathbb{N})$ and $(c_n : n \in \mathbb{N})$ by, $a_n := -1/n$ and $c_n := 1/n$. We may now apply the sandwich theorem since $a_n \leqslant b_n \leqslant c_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = 0$ to obtain that $\lim\limits_{n\to\infty} b_n = 0$. □

We say that the sequence $(a_n : n \in \mathbb{N})$ *diverges to* ∞ if for each $M > 0$ there exists an $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$ and we write: $\lim\limits_{n\to\infty} a_n = \infty$. Likewise we say that $(a_n : n \in \mathbb{N})$ *diverges to* $-\infty$ if for each $M > 0$ there exists an $N \in \mathbb{N}$ such that $a_n < -M$ for all $n > N$ and we write: $\lim\limits_{n\to\infty} a_n = -\infty$.

Note that $\pm\infty$ are not real numbers and so such sequences are **not** convergent.

Example: Show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := \dfrac{n^4}{2n^3+1}$ diverges to ∞.

Answer: First note that $2n^3 + 1 \leqslant 2n^3 + n^3 = 3n^3$ for all $n \in \mathbb{N}$ and so

$$a_n = \frac{n^4}{2n^3+1} \geqslant \frac{n^4}{3n^3} = \frac{n}{3} \quad \text{for all } n \in \mathbb{N}.$$

Now suppose $M > 0$ is given. Let N be any natural number greater than $3M$. Then for each $n > N$, $a_n \geqslant n/3 > N/3 > (3M)/3 = M$. Hence $\lim\limits_{n\to\infty} a_n = \infty$. □

Observation: Let $(a_n : n \in \mathbb{N})$ be a sequence. If it is possible to show that "$a_n \geqslant K \cdot n^p$" for some positive real number K and natural number p, for all $n \in \mathbb{N}$, then $(a_n : n \in \mathbb{N})$ diverges to ∞.

Proof: Let M be any positive real number. If we set N to be any natural number greater than $\sqrt[p]{M/K}$ then, $a_n \geqslant K \cdot n^p > K \cdot N^p > M$ for all $n > N$. □

Monotone convergence

We say that a sequence $(a_n : n \in \mathbb{N})$ is *monotonely increasing* if $a_n \leqslant a_{n+1}$ for all $n \in \mathbb{N}$ and we say that it is *monotonely decreasing* if $a_{n+1} \leqslant a_n$ for all $n \in \mathbb{N}$. Furthermore, we say that a sequence $(a_n : n \in \mathbb{N})$ is *bounded above* if there exists an $M \in \mathbb{R}$ such that $a_n \leqslant M$ for all $n \in \mathbb{N}$ and we say that it is *bounded below* if there exists an $m \in \mathbb{R}$ such that $m \leqslant a_n$ for all $n \in \mathbb{N}$.

Theorem: (Monotone Convergence Theorem) *Let* $(a_n : n \in \mathbb{N})$ *be a monotonely increasing sequence that is bounded above. Then* $\lim\limits_{n\to\infty} a_n$ *exists and equals* $\sup\{a_n : n \in \mathbb{N}\}$.

Proof: Suppose that $\varepsilon > 0$ is given. If we set $L := \sup\{a_n : n \in \mathbb{N}\}$ then $L - \varepsilon$ is **not** an upper bound for the sequence $(a_n : n \in \mathbb{N})$ as L is the least upper bound for this sequence. Hence there exists an $N \in \mathbb{N}$ such that $L-\varepsilon < a_N$. Therefore, $L-\varepsilon < a_N \leqslant a_n \leqslant L < L+\varepsilon$ for all $n > N$, ie: $-\varepsilon < a_n - L < \varepsilon$ for all $n > N$ and so $|a_n - L| < \varepsilon$ for all $n > N$. $\square$

Exercise: Show that each monotonely decreasing sequence that is bounded below is convergent.

Example: Show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_1 := 1$ and $a_{n+1} := \sqrt{1 + a_n}$ is convergent.

Answer: We firstly use induction to prove that the sequence is monotonely increasing. Now $1 = a_1 < \sqrt{2} = \sqrt{1+1} = a_2$ and if we assume that $a_{k-1} \leqslant a_k$, then $a_k = \sqrt{1 + a_{k-1}} \leqslant \sqrt{1 + a_k} = a_{k+1}$; which proves the result. We next prove that the sequence is bounded above. This may also be proved by induction. Now, $a_1 = 1 \leqslant 2$ and if we assume that $a_k \leqslant 2$, then $a_{k+1} = \sqrt{1 + a_k} \leqslant \sqrt{1 + 2} \leqslant 2$; which proves the result. Thus $(a_n : n \in \mathbb{N})$ is monotonely increasing and bounded above by 2. Therefore by the monotone convergence theorem the sequence is convergent. $\square$

Exercise: Calculate the limit of the sequence defined above. *Hint:* use the fact that $a_{n+1}^2 = a_n + 1$, for all $n \in \mathbb{N}$.

Example: Show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_1 := \frac{1}{4}$ and $a_{n+1} := a_n^2 + \frac{1}{4}$ is convergent.

Answer: For each $n \in \mathbb{N}$, $a_{n+1} - a_n = (a_n - 1/2)^2$ and so we have that

$$a_n = a_{n+1} - (a_n - 1/2)^2 \leqslant a_{n+1} \quad \text{for all } n \in \mathbb{N}$$

ie: $(a_n : n \in \mathbb{N})$ is a monotonely increasing sequence. Next we prove, by induction, that this sequence is bounded above by $1/2$. Now, $a_1 = 1/4 \leqslant 1/2$ and if we assume that $a_k \leqslant 1/2$, then $a_{k+1} = a_k^2 + 1/4 \leqslant 1/4 + 1/4 = 1/2$; which proves the result. Hence by the monotone convergence theorem this sequence converges. $\square$

Exercise: Calculate the limit of the sequence defined above. *Hint:* use the fact that $a_{n+1} - a_n = (a_n - 1/2)^2$, for all $n \in \mathbb{N}$.

Theorem: *Let* $(a_n : n \in \mathbb{N})$ *be a monotonely increasing sequence that is* **not** *bounded above then* $(a_n : n \in \mathbb{N})$ *diverges to* ∞.

Proof: Suppose that $M > 0$ is given. Since $(a_n : n \in \mathbb{N})$ is not bounded above there exists an $N \in \mathbb{N}$ such that $a_N > M$ (otherwise, $a_n \leqslant M$ for all $n \in \mathbb{N}$ and $(a_n : n \in \mathbb{N})$ would be bounded above by M). Now since $(a_n : n \in \mathbb{N})$ is monotonely increasing $a_n \geqslant a_N > M$ for all $n > N$. Thus, by definition, $\lim_{n\to\infty} a_n = \infty$. $\square$

So monotonely increasing sequences either converge to a real number L or diverge to ∞, depending upon whether they are bounded or not.

Completeness

We say that a sequence $(a_n : n \in \mathbb{N})$ is a *Cauchy* sequence if for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $m, n > N$.

Exercise: Show that each Cauchy sequence is bounded.

Exercise: Let $(a_n : n \in \mathbb{N})$ be a bounded sequence. Show that for each $N \in \mathbb{N}$,

$$\sup\{a_n : n \geqslant N\} - \inf\{a_n : n \geqslant N\} \leqslant \sup\{|a_n - a_m| : m, n \geqslant N\}$$

Hint: it is sufficient to show that for each $\varepsilon > 0$,

$$\sup\{a_n : n \geqslant N\} - \inf\{a_n : n \geqslant N\} \leqslant \sup\{|a_n - a_m| : m, n \geqslant N\} + \varepsilon.$$

Theorem: *A sequence of real numbers is convergent if, and only if, it is Cauchy.*

Proof: We will prove first that convergent sequences are Cauchy (this is the easy direction). To this end, let $(a_n : n \in \mathbb{N})$ be any convergent sequence and ε be any positive real number. Now since $(a_n : n \in \mathbb{N})$ is convergent we may choose an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon/2$ for all $n > N$ and so $|a_n - a_m| \leqslant |a_n - L| + |a_m - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $m, n > N$. This completes this half of the proof. Converse: Suppose that $(c_n : n \in \mathbb{N})$ is a Cauchy sequence. For each $n \in \mathbb{N}$, let $a_n := \inf\{c_k : k \geqslant n\}$ and $b_n := \sup\{c_k : k \geqslant n\}$. Then since $(c_n : n \in \mathbb{N})$ is bounded (see above),

$$-\infty < a_1 \leqslant a_n \leqslant b_n \leqslant b_1 < \infty \quad \text{for all } n \in \mathbb{N}.$$

Now $(a_n : n \in \mathbb{N})$ is a monotonely increasing sequence that is bounded above by b_1 and $(b_n : n \in \mathbb{N})$ is a monotonely decreasing sequence that is bounded below by a_1. Hence by the monotone convergence theorem both $A := \lim_{n\to\infty} a_n$ and $B := \lim_{n\to\infty} b_n$ exist. Moreover, if $n > N$ then,

$$a_N \leqslant a_n \leqslant b_n \leqslant b_N \quad \text{and so} \quad 0 \leqslant b_n - a_n \leqslant b_N - a_N.$$

By applying the previous exercise and taking the limit as $n \to \infty$ we see that for any $N \in \mathbb{N}$

$$0 \leqslant B - A \leqslant b_N - a_N \leqslant \sup\{|c_n - c_m| : m, n \geqslant N\}.$$

Therefore, since $(c_n : n \in \mathbb{N})$ is a Cauchy sequence, $0 \leqslant B - A < \varepsilon$ for each $\varepsilon > 0$ and so $A = B$. The result now follows from the sandwich theorem since $a_n \leqslant c_n \leqslant b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. $\square$

This result is important as it enables us to show that a given sequence is convergent without, a priori, knowing the limit.

Exercises

1. Let a be any real number. Show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := a$ for all $n \in \mathbb{N}$ is convergent. (What is the limit ?).

2. Prove, using an ε - N argument, that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := n/(n+1)$ for all $n \in \mathbb{N}$ converges to 1.

3. Show, using the sum, product and division rules for limits, that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := (4n^3 + 2n + 1)/(n+1)^3$ converges to 4.

4. Let $0 < r < 1$. Show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := r^n$ is monotonely decreasing and bounded below by 0. Hence show that $\lim\limits_{n\to\infty} a_n = 0$. *Hint:* $a_{n+1} := r \cdot a_n$ for all $n \in \mathbb{N}$.

5. Let $g : (0, \infty) \to \mathbb{R}$ be the function defined by, $g(x) := \frac{1}{2}(x + 4/x)$.

(a) Show that $g(x) \geqslant 2$ for all $x > 0$. *Hint:* Expand $\frac{1}{2}(\sqrt{x} - 2/\sqrt{x})^2$.

(b) Let $(a_n : n \in \mathbb{N})$ be the sequence defined by, $a_{n+1} := \frac{1}{2}(a_n + 4/a_n)$ for all $n \in \mathbb{N}$ and $a_1 := 4$. By using the result from part (a) show that $a_n \geqslant 2$ for all $n \in \mathbb{N}$.

(c) Prove that the sequence $(a_n : n \in \mathbb{N})$ is monotonely decreasing.

(d) Hence deduce that the sequence $(a_n : n \in \mathbb{N})$ is convergent. Also, calculate the limit of this sequence.

6. Consider the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := (2n - 7)/(3n + 2)$ for all $n \in \mathbb{N}$.

(a) Prove that the sequence $(a_n : n \in \mathbb{N})$ is bounded above by $2/3$.

(b) Verify that the sequence $(a_n : n \in \mathbb{N})$ is monotonely increasing by showing that $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N}$.

(c) Deduce that the sequence $(a_n : n \in \mathbb{N})$ converges and calculate its limit.

7. Consider the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := \sqrt{n}/(n+1)$ for all $n \in \mathbb{N}$.

(a) Show that the sequence $(a_n : n \in \mathbb{N})$ is convergent by showing that it is monotonely decreasing and bounded below.

(b) Verify your answer to part (a) by applying the sandwich theorem to suitably chosen sequences.

8. Let $a > 0$. Show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := a^n/n!$ for all $n \in \mathbb{N}$ converges to 0. *Hint:* show that "eventually" the sequence $(a_n : n \in \mathbb{N})$ is monotonely decreasing and bounded below by 0. To calculate $\lim\limits_{n\to\infty} a_n$ it may be helpful to use the fact that $a_{n+1} = [a/(n+1)] \cdot a_n$ for all $n \in \mathbb{N}$.

9. Prove that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := \sqrt[n]{n!}$ for all $n \in \mathbb{N}$, diverges to ∞. Note: the result from **8** may be helpful.

10. Show that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_1 := 1$ and $a_{n+1} := a_n^2 + \frac{1}{4}$ is monotonely increasing and diverging to ∞. Note: $a_{n+1} - a_n = (a_n - 1/2)^2$ for all $n \in \mathbb{N}$.

11. Suppose that a sequence $(a_n : n \in \mathbb{N})$ has the property that for some $0 < r < 1$, $|a_{n+2} - a_{n+1}| \leqslant r \cdot |a_{n+1} - a_n|$ for all $n \in \mathbb{N}$.

(a) Prove by induction that $|a_{n+2} - a_{n+1}| \leqslant r^n \cdot |a_2 - a_1|$ for all $n \in \mathbb{N}$.

(b) Show that for each $1 < m < n$,

$$|a_n - a_m| \leqslant \sum_{k=m}^{n-1} |a_{k+1} - a_k| \leqslant |a_2 - a_1| \sum_{k=m}^{n-1} r^{k-1} \leqslant |a_2 - a_1| \sum_{k=m}^{\infty} r^{k-1} = |a_2 - a_1| \frac{r^{m-1}}{(1-r)}.$$

(c) Deduce that the sequence $(a_n : n \in \mathbb{N})$ is Cauchy and hence convergent.

(d) Let $(a_n : n \in \mathbb{N})$ be the sequence defined by, $a_1 := 1$, $a_2 := 3$ and $a_{n+2} := \frac{1}{2}(a_{n+1} + a_n)$ for all $n \in \mathbb{N}$. Show that this sequence is convergent.

12. Suppose that $a < b$ and $f : [a, b] \to [a, b]$. If $0 < r < 1$ and $|f(x) - f(y)| \leqslant r|x - y|$ for all $x, y \in [a, b]$ show that;

(a) any sequence $(a_n : n \in \mathbb{N})$ defined by, $a_1 \in [a, b]$ and $a_{n+1} := f(a_n)$ for all $n \in \mathbb{N}$ satisfies the inequality $|a_{n+2} - a_{n+1}| \leqslant r \cdot |a_{n+1} - a_n|$ for all $n \in \mathbb{N}$;

(b) the sequence $(a_n : n \in \mathbb{N})$ is convergent;

(c) if $a_\infty := \lim_{n \to \infty} a_n$ then $f(a_\infty) = a_\infty$. *Hint:* Show that for each $n \in \mathbb{N}$, $|f(a_\infty) - a_\infty| \leqslant |f(a_\infty) - a_{n+1}| + |a_{n+1} - a_\infty| = |f(a_\infty) - f(a_n)| + |a_{n+1} - a_\infty|$.

13. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by, $f(x) := (1/5)(x^3 + x + 1)$.

(a) Show that f maps the interval $[0, 1]$ into $[0, 1]$.

(b) Using the result from **12** deduce that the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_1 := 1/2$ and $a_{n+1} := f(a_n)$ is convergent.

14. Show that the sequence $(a_n : n \in \mathbb{N})$ defined by $a_1 := 0$ and $a_{n+1} := 2a_n(1 - a_n) + 1/5$ is convergent.

Infinite series

Introduction

Let $(a_n : n \in \mathbb{N})$ be a sequence. For each $n \in \mathbb{N}$, we define the *nth partial sum* of $(a_n : n \in \mathbb{N})$ to be: $s_n := a_1 + a_2 + \ldots a_n$. By the *series* generated by $(a_n : n \in \mathbb{N})$ we mean the sequence $(s_n : n \in \mathbb{N})$ of partial sums of $(a_n : n \in \mathbb{N})$ (ie: a series is a sequence). We say that the series $(s_n : n \in \mathbb{N})$ generated by a sequence $(a_n : n \in \mathbb{N})$ is *convergent* if the sequence $(s_n : n \in \mathbb{N})$ of partial sums converges. If the series $(s_n : n \in \mathbb{N})$ is convergent then we call its limit the *sum* of the series and we write: $\lim\limits_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$.

A series that is **not** convergent is called *divergent*, ie: each series is either convergent or divergent. It has become traditional to use the notation $\sum_{n=1}^{\infty} a_n$ to represent both the series $(s_n : n \in \mathbb{N})$ generated by the sequence $(a_n : n \in \mathbb{N})$ and the sum $\lim_{n\to\infty} s_n$. However, this ambiguity in the notation should not lead to any confusion, provided that it is always made clear that the convergence of the series **must** be established.

Important: Although traditionally series have been (and still are) represented by expressions such as $\sum_{n=1}^{\infty} a_n$ they are, technically, sequences of partial sums. So if somebody comes up to you in a supermarket and asks you what a series is - you answer "it is a sequence of partial sums" **not** "it is an expression of the form: $\sum_{n=1}^{\infty} a_n$."

Example: For $0 < r < \infty$, we may define the sequence $(a_n : n \in \mathbb{N})$ by, $a_n := r^{n-1}$ for each $n \in \mathbb{N}$. Then the partial sums of this sequence are: $s_n := 1 + r + r^2 + \ldots r^{n-1}$. Therefore, $r \cdot s_n = r + r^2 + r^3 + \ldots r^n$ and so $(1 - r)s_n = s_n - rs_n = 1 - r^n$. Hence if $r \neq 1$ then, $s_n = (1 - r^n)/(1 - r)$. Moreover, for $0 < r < 1$, $(s_n : n \in \mathbb{N})$ is monotonely increasing and bounded above by $1/(1 - r)$ since, $(1 - r^n) < 1$ for all $n \in \mathbb{N}$. Hence the limit of the sequence $(s_n : n \in \mathbb{N})$ exists. Furthermore, since $s_{n+1} = r \cdot s_n + 1$ for all $n \in \mathbb{N}$ we have that $L = rL + 1$, ie: $L = 1/(1 - r)$, where $L := \lim\limits_{n\to\infty} s_n$. $\square$

The series in the above example is called the *geometric series* with *ratio* r.

Example: For each $0 < p < \infty$, let $(a_n : n \in \mathbb{N})$ be the sequence defined by, $a_n := 1/n^p$ for all $n \in \mathbb{N}$. Then the series generated by $(a_n : n \in \mathbb{N})$ is called a *p-series*. It will be shown later that a p-series is convergent if, and only if, $1 < p < \infty$. The case $p = 1$ has special significance and is called the *harmonic* series.

Exercise: Let $(a_n : n \in \mathbb{N})$ be the sequence defined by, $a_n := (-1)^n$ for all $n \in \mathbb{N}$. Show that the series generated by $(a_n : n \in \mathbb{N})$ is divergent.

Example: Calculate the sum of the series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

Answer: By using partial fractions we see that,

$$\begin{aligned} s_n &= \sum_{j=1}^{n} \frac{1}{j(j+1)(j+2)} \\ &= \frac{1}{2}\sum_{j=1}^{n}\left(\frac{1}{j(j+1)} - \frac{1}{(j+1)(j+2)}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{12} + \cdots + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) \end{aligned}$$

and so $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = 1/4.$ $\square$

Any series of the form: $\sum_{n=1}^{\infty}(a_n - a_{n+1})$, for some sequence $(a_n : n \in \mathbb{N})$ is called a *telescoping* series and so we see that the previous example is in fact a telescoping series.

Exercise: Let $(a_n : n \in \mathbb{N})$ be a sequence. Show that the series $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges if, and only if, the sequence $(a_n : n \in \mathbb{N})$ converges. *Hint:* Show that the nth-partial sum of this series equals: $a_1 - a_{n+1}$.

Exercise: Express the sum of the series, $\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+a+1)}$ in terms of a.

Theorem: *If* $\sum_{n=1}^{\infty} a_n$ *is a convergent series then* $\lim_{n\to\infty} a_n = 0$.

Proof: Let s_n denote the nth partial sum of the series. Then $s_n - s_{n-1} = a_n$ for all $n \in \mathbb{N}$ and so by taking the limit of both sides of this equation as, $n \to \infty$, we see that,

$$0 = \left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n\right) = \left(\lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}\right) = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} a_n \qquad \square$$

Warning: The fact that $\lim_{n\to\infty} a_n = 0$ does **not** guarantee that $\sum_{n=1}^{\infty} a_n$ is convergent.

Example: The harmonic series, $\sum_{n=1}^{\infty} 1/n$ is divergent.

Proof: We prove this result by induction. Let,

$$s_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{(n-1)} + \frac{1}{n} \quad \text{and} \quad P(n) := \text{“}s_{2^n} \geqslant 1 + n/2\text{”}$$

Step 1. $P(1) =$ “$s_{2^1} \geqslant 1 + 1/2$”; which is true. Therefore $P(n)$ is true for the case $n = 1$.
Step 2. Suppose that $P(k)$ is true, ie: suppose that $s_{2^k} \geqslant 1 + k/2$, then,

$$s_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{k+1}}$$

$$
\begin{aligned}
&= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{2^k}\right)+\left(\frac{1}{2^k+1}+\cdots+\frac{1}{2^{k+1}}\right)\\
&\geqslant (1+k/2)+\left(\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}\cdots+\frac{1}{2^{k+1}}\right)\\
&= (1+k/2)+1/2=1+(k+1)/2.
\end{aligned}
$$

Therefore $P(k+1)$ is true.

Step 3. Hence by induction, $P(n)$ is true for all $n \in \mathbb{N}$, ie: $s_{2^n} \geqslant 1+n/2$ for all $n \in \mathbb{N}$. Now $(s_n : n \in \mathbb{N})$ is a monotonely inceasing sequence (why ?) that is **not** bounded above. Hence, $\sum\limits_{n=1}^{\infty} 1/n = \lim\limits_{n\to\infty} s_n$ diverges to ∞. Note however that, $\lim\limits_{n\to\infty} 1/n = 0$. □

The following theorem enables us to show that certain series converge without having to resort to using ε's and N's.

Theorem: *Let* $\sum\limits_{n=1}^{\infty} a_n$ *and* $\sum\limits_{n=1}^{\infty} b_n$ *be convergent series then;*

(i) $\sum\limits_{n=1}^{\infty}(a_n+b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$;

(ii) $\sum\limits_{n=1}^{\infty} ca_n = c\cdot\left(\sum\limits_{n=1}^{\infty} a_n\right)$.

Proof: Exercise.

Series of non-negative terms

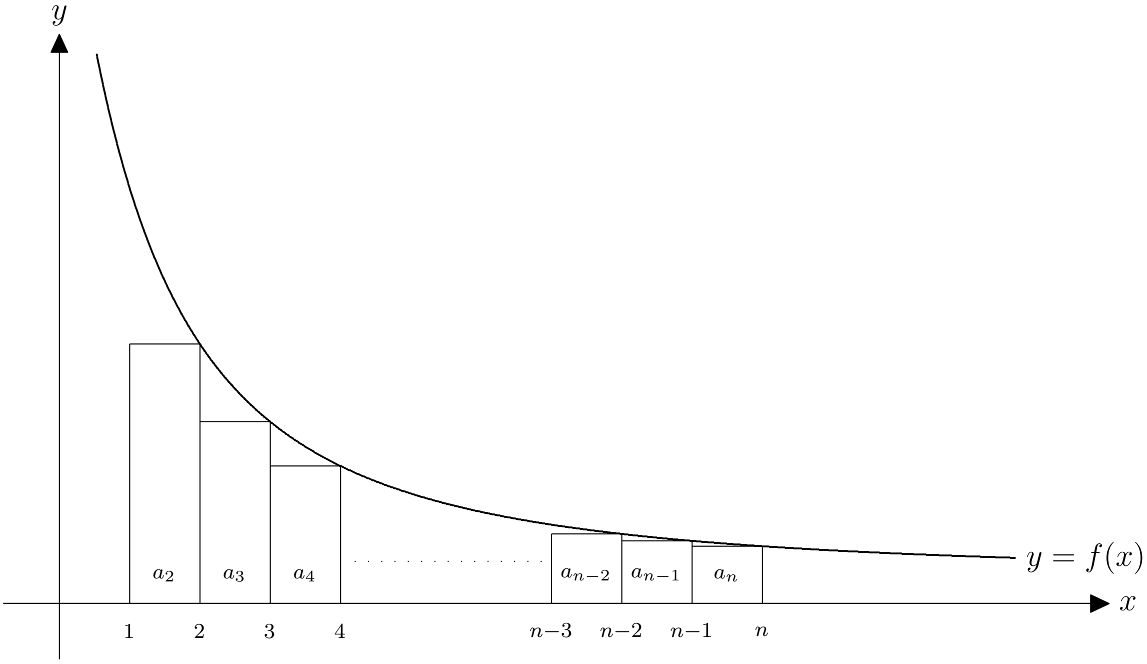
If $(a_n : n \in \mathbb{N})$ is a sequence with non-negative terms (ie: $0 \leqslant a_n$ for all $n \in \mathbb{N}$) then the series $\sum_{n=1}^{\infty} a_n$ is called a *series of non-negative terms.* Importantly for such series the partial sums $s_n := \sum_{j=1}^{n} a_j$ are monotonely increasing since $s_{n+1} - s_n = \sum_{j=1}^{n+1} a_j - \sum_{j=1}^{n} a_j = a_{n+1} \geqslant 0$.

Hence to show that a series of non-negative terms is convergent it is sufficient (and necessary) to show that the series is bounded above.

We now look at some tests for convergence of series with non-negative terms. The first test we consider is the integral test.

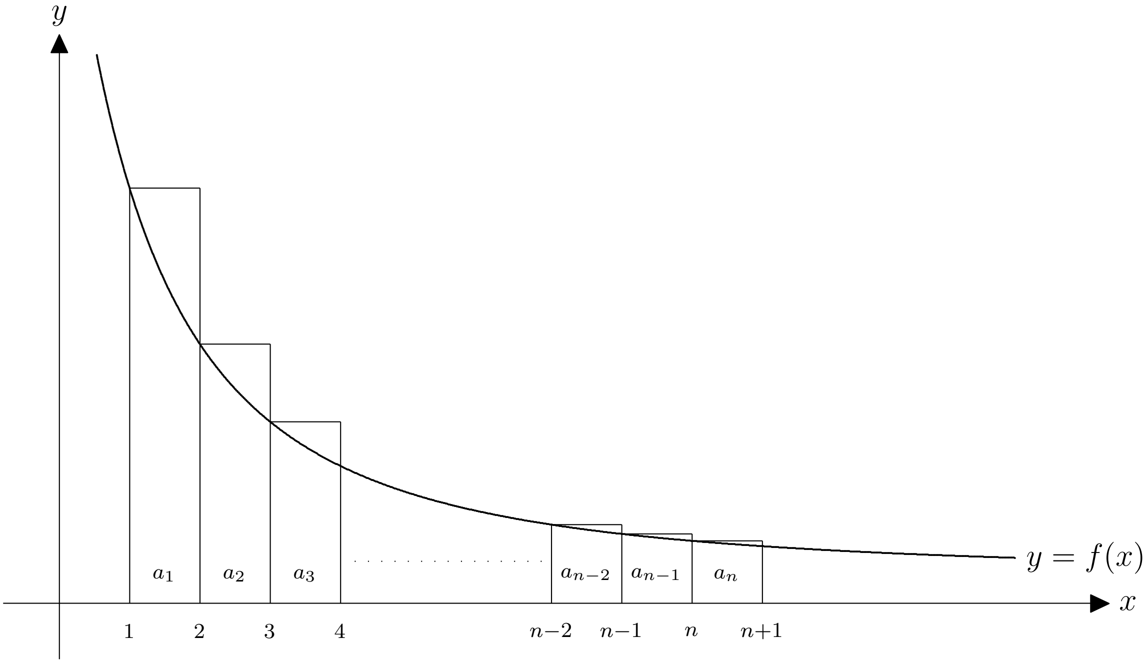
Theorem: (Integral Test) *Let* $(a_n : n \in \mathbb{N})$ *be a sequence of non-negative terms and let* $f : [1,\infty) \to \mathbb{R}$ *be a monotonely decreasing continuous function such that* $f(n) = a_n$ *for all* $n \in \mathbb{N}$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *converges if, and only if, the integral* $\int_1^{\infty} f(x)dx = \lim_{N\to\infty}\int_1^N f(x)dx$ *converges.*

Proof: In the next figure, the area of the rectangles is: $a_2+a_3+\cdots+a_{n-1}+a_n$ and is clearly no more than the area under the curve and so, $a_2+a_3+\cdots+a_{n-1}+a_n \leqslant \displaystyle\int_1^n f(x)dx$.



Thus, $s_n \leqslant a_1 + \int_1^n f(x)dx$.

In the next figure we see that the area of the rectangles is: $a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$ and so $\int_1^{n+1} f(x)dx \leqslant a_1 + a_2 + \cdots + a_{n-1} + a_n$



Hence, $\int_1^{n+1} f(x)dx \leqslant s_n \leqslant a_1 + \int_1^n f(x)dx$.

By taking the limit as, $n \to \infty$, we see that the series and improper integral are either both convergent or both divergent. Note also that the last inequality gives us both an upper and lower bound for the partial sum s_n. $\Box$

Example: Show that the p-series, $\displaystyle\sum_{n=1}^{\infty} 1/n^p$ is convergent if, and only if, $1 < p < \infty$.

Answer: Suppose that $0 < p < \infty$. Then the function $f : [1, \infty) \to \mathbb{R}$ defined by, $f(x) := 1/x^p$ satisfies the hypotheses of the integral test. Moreover, for $p \neq 1$ we have that:

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{n\to\infty} \int_1^n x^{-p} dx = \lim_{n\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^n = \lim_{n\to\infty} \left[\frac{n^{1-p}-1}{1-p}\right] = \begin{cases} \dfrac{1}{p-1} & p > 1, \\ \infty & 0 < p < 1. \end{cases}$$

For $p = 1$, (ie: the harmonic series) we have that:

$$\lim_{n\to\infty} \int_1^n x^{-1} dx = \lim_{n\to\infty} [\log_e(n) - \log_e(1)] = \infty.$$

Thus the integral test shows that a p-series is convergent if, and only if, $1 < p < \infty$. $\Box$

Let $(a_n : n \in \mathbb{N})$ be a sequence. Then for each $n \in \mathbb{N}$, we define the *N-th tail* of the series $(s_n : n \in \mathbb{N})$, generated by $(a_n : n \in \mathbb{N})$, to be the the series generated by the N-th tail of the sequence $(a_n : n \in \mathbb{N})$. Traditionally, the N-th tail of $\sum_{n=1}^{\infty} a_n$ is denoted $\sum_{n=N+1}^{\infty} a_n$.

Exercise: Show that a series $\sum_{n=1}^{\infty} a_n$ is convergent if, and only if, for some $N \in \mathbb{N}$ the N-th tail of $\sum_{n=1}^{\infty} a_n$ is convergent, ie: if, and only if, the "tail" of the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Theorem: (Comparison Test) *Let $(a_n : n \in \mathbb{N})$ and $(b_n : n \in \mathbb{N})$ be sequences. If $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges then so does $\sum_{n=1}^{\infty} a_n$.*

Proof: For each $n \in \mathbb{N}$, let $s_n := \sum_{j=1}^{n} a_j$ and $t_n := \sum_{j=1}^{n} b_j$. Then $(s_n : n \in \mathbb{N})$ is monotonely increasing and $s_n \leqslant t_n \leqslant \sup\{t_k : k \in \mathbb{N}\} = \lim_{k\to\infty} t_k = \sum_{k=1}^{\infty} b_k$ for each $n \in \mathbb{N}$. Hence by the monotone convergence theorem the sequence $(s_n : n \in \mathbb{N})$, of partial sums is convergent (ie: $\sum_{n=1}^{\infty} a_n$ is convergent). $\Box$

Example: Show that the series $\sum\limits_{n=1}^{\infty} |\sin(n)|/n^2$ is convergent.

Answer: First we observe that, $0 \leqslant |\sin(n)|/n^2 \leqslant 1/n^2$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} 1/n^2$ is a convergent p-series. Therefore, by the comparison test, $\sum_{n=1}^{\infty} |\sin(n)|/n^2$ is convergent.

Theorem: (Limit Comparison Test) *Let $(a_n : n \in \mathbb{N})$ and $(b_n : n \in \mathbb{N})$ be sequence of non-negative terms. If $\lim\limits_{n\to\infty} (a_n/b_n)$ exists and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof: Since $\lim_{n\to\infty}(a_n/b_n)$ exists there exists an $M > 0$ such that $a_n/b_n \leqslant M$ for all $n \in \mathbb{N}$ (since convergent sequences are bounded). Now if we define the sequence $(c_n : n \in \mathbb{N})$ by, $c_n := M \cdot b_n$ for all $n \in \mathbb{N}$, then we have that, $0 \leqslant a_n \leqslant c_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n$ is convergent. Therefore, the result follows from the comparison test. $\Box$

Example: Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(\sqrt{n^2+1} - n)}{n^2}$ convergent ?

Answer: Yes - and here is the reason. Consider the sequences $(a_n : n \in \mathbb{N})$ and $(b_n : n \in \mathbb{N})$ defined by, $a_n := (\sqrt{n^2+1} - n)/n^2$ for all $n \in \mathbb{N}$ and $b_n := 1/n^2$ for all $n \in \mathbb{N}$. Then $(a_n/b_n) = (\sqrt{n^2+1} - n)$ for all $n \in \mathbb{N}$ and so $\lim\limits_{n\to\infty}(a_n/b_n) = \lim\limits_{n\to\infty}(\sqrt{n^2+1} - n) = 0$. The result now follows from the limit comparison test. □

Exercise: Show that the series $\displaystyle\sum_{n=1}^{\infty} \log_e(n)/n^3$ is convergent.

Theorem: (Cauchy Condition for Series)*A series* $\sum_{n=1}^{\infty} a_n$ *is convergent if, and only if, for each* $\varepsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that for each* $m > n > N$, $|a_{n+1}+a_{n+2}+ \ldots a_m| < \varepsilon$.

Proof: Let $(s_n : n \in \mathbb{N})$ be the series generated by $(a_n : n \in \mathbb{N})$. Now the series $\sum_{n=1}^{\infty} a_n$ is convergent if, and only if, $\lim_{n\to\infty} s_n$ exists, (by definition). Therefore the result follows from the fact that $\lim\limits_{n\to\infty} s_n$ exists if, and only if, the sequence $(s_n : n \in \mathbb{N})$ is Cauchy. □

A series $\displaystyle\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem: Each absolutely convergent series is convergent.

Proof: Suppose that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, ie: suppose that $\sum_{n=1}^{\infty} |a_n|$ is convergent. Define the sequence $(b_n : n \in \mathbb{N})$ by, $b_n := a_n + |a_n|$ for all $n \in \mathbb{N}$. Now since, $0 \leqslant b_n \leqslant 2 \cdot |a_n|$ for all $n \in \mathbb{N}$, we have that the series $\sum_{n=1}^{\infty} b_n$ converges by the comparison test. Furthermore, since $a_n = b_n - |a_n|$ for all $n \in \mathbb{N}$, we see that in actual fact the series $\sum_{n=1}^{\infty} a_n$ is convergent; which is the desired result. □

Warning: The converse of the previous theorem is **not** true, ie: there do exist convergent series that are not absolutely convergent.

A convergent series that is **not** absolutely convergent is called *conditionally convergent.*

Exercise: Give an alternative proof of the previous theorem by;

(a) showing that if $(s_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ are the series generated by the sequences $(a_n : n \in \mathbb{N})$ and $(|a_n| : n \in \mathbb{N})$ respectively. Then for each $m, n \in \mathbb{N}$, $|s_n - s_m| \leqslant |t_n - t_m|$;

(b) applying the Cauchy condition for convergence of a series.

Theorem: (Ratio Test) *Let* $(a_n : n \in \mathbb{N})$ *be a sequence of non-zero real numbers; (i) if there is a positive number* $r < 1$ *and a natural number* $N \in \mathbb{N}$ *such that* $|a_{n+1}|/|a_n| < r$ *for all* $n > N$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent; (ii) if there exists a number* $N \in \mathbb{N}$ *such that* $|a_{n+1}|/|a_n| \geqslant 1$ *for all* $n > N$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof: (i) It follows by an easy induction argument that $|a_{N+m}| \leqslant r^m |a_N|$ for all $m \in \mathbb{N}$. Hence if we define a convergent geometric series $\sum_{n=1}^{\infty} b_n$ by, $b_n := r^{n-N}|a_N|$ for all $n \in \mathbb{N}$ then $|a_n| \leqslant b_n$ for all $n \geqslant N$ and so the N-th tail of $\sum_{n=1}^{\infty} |a_n|$ converges by the comparison test. This however, implies that $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(ii) By again using induction we see that $|a_{N+n}| \geqslant |a_N| > 0$ for all $n \in \mathbb{N}$. Therefore, since the terms of $(a_n : n \in \mathbb{N})$ do not converge to 0 the series must be divergent. □

Exercise: Let $(a_n : n \in \mathbb{N})$ be a sequence of non-zero real numbers. Show that; (i) if $\lim_{n\to\infty} |a_{n+1}|/|a_n| < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent; (ii) if $\lim_{n\to\infty} |a_{n+1}|/|a_n| > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note that in the previous example we cannot make any deductions in the case when $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1$. For example, $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1$ if $a_n := 1/n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1$ if $a_n := 1/n^2$ for all $n \in \mathbb{N}$. However, in the first case the series diverges while in the second case the series converges.

Let σ be a function from $\mathbb{N}$ into $\mathbb{N}$. We call σ a *permutation* of $\mathbb{N}$ if; (i) σ is 1-to-1; (ii) σ is onto, ie: a permutation of $\mathbb{N}$ is a bijection on $\mathbb{N}$.

Exercise: Show that the set $S_{\mathbb{N}}$ of all permutations on $\mathbb{N}$, under the (binary) operation of composition forms a group.

A series $\sum_{n=1}^{\infty} b_n$ is called a *rearrangement* of a series $\sum_{n=1}^{\infty} a_n$ if there exists a permutation σ of $\mathbb{N}$ such that $b_n := a_{\sigma(n)}$ for all $n \in \mathbb{N}$. Intuitively, this is saying that a series $\sum_{n=1}^{\infty} b_n$ is a "rearrangement" of another series $\sum_{n=1}^{\infty} a_n$ if it is obtained from $\sum_{n=1}^{\infty} a_n$ by just summing the terms up in a different order. Counter intuitively, by changing the order of the summation you may be able to change the value of the sum. In fact one can obtain the following **very** counter intuitive result.

Theorem: (Riemann)*Let $\sum_{n=1}^{\infty} a_n$ be any conditionally convergent series. Then for each $r \in \mathbb{R}$ there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)} = r$, ie: by changing the order of summation you can get the sum to equal anything you like.*

Proof: The idea of the proof is simple: we sum up the positive terms until we obtain a partial sum exceeding r, then we sum up the negative terms from the series until we obtain a partial sum less than r. We then keep repeating this process indefinitely. Since $\lim_{n\to\infty} a_n = 0$, it is not difficult to see that this process produces a rearrangement that converges to r. $\square$

Such pathological behaviour as described above is rather undesirable. Fortunately, absolutely convergent series are much better behaved.

Theorem: (Rearrangement Theorem) *For any permutation σ of $\mathbb{N}$ and absolutlely convergent series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}$.*

Proof: Suppose that $\sum_{n=1}^{\infty} a_n = L$ and σ is a permutation of $\mathbb{N}$; we will show that $\sum_{n=1}^{\infty} a_{\sigma(n)} = L$. Let ε be any positive real number and let us choose $N \in \mathbb{N}$ such that if $s_N := a_1 + a_2 + \ . \ . \ . \ a_N$, then $|s_N - L| < \varepsilon/2$ and $\sum_{j=N+1}^{n} |a_j| < \varepsilon/2$ for all $n > N$. Now let $M \in \mathbb{N}$ be chosen so that all of the terms $a_1, a_2, \ . \ . \ a_N$ are contained in the sum $t_M := a_{\sigma(1)} + a_{\sigma(2)} + \ . \ . \ . \ a_{\sigma(M)}$. It follows that if $k \geqslant M$ then $t_k - s_N$ consists of a finite sum of terms a_j with $j > N$ and so for some $n_k > N$ we have $|t_k - s_N| \leqslant \sum_{j=N+1}^{n_k} |a_j| < \varepsilon/2$. It now follows that $|t_k - L| \leqslant |t_k - s_N| + |s_N - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $k \geqslant M$. This shows that $\sum_{n=1}^{\infty} a_{\sigma(n)} = L$. $\square$

Alternating series

Any series of the form: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, with $0 \leqslant a_n < \infty$ for all $n \in \mathbb{N}$, is called an *alternating series.*

So far in these notes we have only considered tests that imply absolute convergence. However, below we shall present a test for conditional convergence. In general, tests for conditional convergence are much more delicate than tests for absolute convergence.

Lemma: *Let* $(a_n : n \in \mathbb{N})$ *be a sequence. If* $0 \leqslant a_{n+1} \leqslant a_n$ *for all* $n \in \mathbb{N}$ *then* $|s_m - s_n| \leqslant a_{n+1}$ *for all* $m > n$*, where* $s_k := \sum_{j=1}^{k}(-1)^{j+1}a_j$ *for all* $k \in \mathbb{N}$*.*

Proof: Suppose that n is even, then $s_{n+2m+1} - s_n = a_{n+1} + (-a_{n+2} + a_{n+3}) + (-a_{n+4} + a_{n+5}) + \ldots (-a_{n+2m} + a_{n+2m+1}) \leqslant a_{n+1}$ for all $m \geqslant 0$ and since, $s_{n+2m} \leqslant s_{n+2m+1} = s_{n+2m} + a_{n+2m+1}$ for all $m \geqslant 0$, $s_{n+2m} - s_n \leqslant a_{n+1}$ for all $m \geqslant 0$. Therefore, $s_k - s_n \leqslant a_{n+1}$ for all $k \geqslant n$. Now, $s_{n+2m} - s_n = (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \ \ldots\ (a_{n+2m-1} - a_{n+2m}) \geqslant 0$ for all $m \geqslant 0$ and since $s_{n+2m+1} \geqslant s_{n+2m}$ for all $m \geqslant 0$, $s_{n+2m+1} - s_n \geqslant 0$ for all $m \geqslant 0$, ie: $s_k - s_n \geqslant 0$ for all $k \geqslant n$. Hence, $0 \leqslant s_k - s_n \leqslant a_{n+1}$ for all $k \geqslant n$. A similar argument shows that for n odd, $-a_{n+1} \leqslant s_k - s_n \leqslant 0$ for all $k \geqslant n$. So finally we have the desired result that $|s_k - s_n| \leqslant a_{n+1}$ for all $k \geqslant n$. □

Exercise: Let $(a_n : n \in \mathbb{N})$ be a convergent sequence. Show that $\left|\lim\limits_{n\to\infty} a_n\right| = \lim\limits_{n\to\infty} |a_n|$.

Theorem: (Alternating Series Test) *If* $(a_n : n \in \mathbb{N})$ *is a sequence with* $0 < a_{n+1} \leqslant a_n$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} a_n = 0$*, then* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n := L$ *converges and* $|L - s_n| \leqslant a_{n+1}$ *for all* $n \in \mathbb{N}$*. Here* $s_n := \sum_{k=1}^{n}(-1)^{k+1}a_k$*.*

Proof: We claim that the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ satisfies the Cauchy condition. To see this, let us suppose that $\varepsilon > 0$ is given. Then since $\lim_{n\to\infty} a_n = 0$ there exists an $N \in \mathbb{N}$ such that $0 < a_n < \varepsilon$ for all $n > N$. Hence for any, $m > n > N$, $|s_m - s_n| \leqslant a_{n+1} \leqslant a_n < \varepsilon$. This completes the proof of the claim. It now follows that the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n := L$ is convergent. Next, let us fix $n \in \mathbb{N}$ and let $m > n$. Then $|s_m - s_n| \leqslant a_{n+1}$ and by taking the limit as, $m \to \infty$, we see that,
$|L - s_n| = |\lim\limits_{m\to\infty} s_m - s_n| = |\lim\limits_{m\to\infty}(s_m - s_n)| = \lim\limits_{m\to\infty} |s_m - s_n| \leqslant a_{n+1}$. □

Exercise: Provide an alternative (and simpler) proof of the alternating series test by;

(a) showing that if $(a_n : n \in \mathbb{N})$ is a sequence such that $\lim_{m\to\infty} a_{2m} = \lim_{m\to\infty} a_{2m+1}$ then $\lim_{n\to\infty} a_n$ exists;

(b) showing that the sequence $(s_{2n} : n \in \mathbb{N})$ is convergent. *Hint:* it maybe helpful to consider the following equations.
$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2})$ and $s_{2n} = a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leqslant a_1$

(c) showing that the sequence $(s_{2n+1} : n \in \mathbb{N})$ is convergent and $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1}$. Note: $s_{2n+1} = s_{2n} + a_{2n+1}$ for all $n \in \mathbb{N}$.

(d) putting steps (a), (b) and (c) together to show that the series is convergent.

Example: Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent.

Answer: This series is convergent by the alternating series test, but it is **not** absolutely convergent. This series has special significance as it is the simplest example of a conditionally convergent series. In fact this series has its own name: the *alternating harmonic* series. □

Power series

A series of the form: $\sum_{n=0}^{\infty} a_n x^n \equiv a_0 + a_1 x + a_2 x^2 + \ . \ . \ . \ a_n x^n + \ . \ .$ where $a_n \in \mathbb{R}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}$, is called a *power series* (in x). For each fixed $x \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} a_n x^n$ may or may not converge, ie: the limit, $\lim_{n\to\infty} \left(\sum_{k=0}^{n} a_k x^k\right)$ may or may not exist. By investigating the convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ we mean: determine the set $\{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n$ is convergent$\}$. It is obvious that every power series converges at $x = 0$ to a_0. However, we have the following important theorem concerning the set $\{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n$ is convergent$\}$.

Theorem: *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges when $x = x_0$ then it is absolutely convergent for every $x \in \mathbb{R}$ such that $|x| < |x_0|$.*

Proof: For any $x \in \mathbb{R}$ such that $|x| < |x_0|$ we have that $|a_n x^n| = |a_n x_0^n| \cdot |x|^n/|x_0|^n$. Since $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent, $\{a_n x_0^n : n \in \mathbb{N} \cup \{0\}\}$ is bounded; that is, there exists an $M > 0$ such that $|a_n x_0^n| < M$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $|a_n x^n| \leqslant M|x|^n/|x_0|^n$ for all $n \in \mathbb{N} \cup \{0\}$. However, $\sum_{n=0}^{\infty} |x|^n/|x_0|^n$ is a geometric series with ratio $|x|/|x_0| < 1$. Therefore this series converges and by the comparison test $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent; that is, $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for $|x| < |x_0|$. $\square$

For a given power series $\sum_{n=0}^{\infty} a_n x^n \equiv a_0 + a_1 x + a_2 x^2 + \ . \ . \ a_n x^n + \ . \ .$ there are three possibilities for the set: $\{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n$ is convergent$\}$ either;

(i) the series is only convergent at $x = 0$;

(ii) the series is convergent for all $x \in \mathbb{R}$;

(iii) there exists a $\rho > 0$ such that the series is absolutely convergent for $|x| < \rho$ and divergent for $|x| > \rho$. (the case $|x| = \rho$ depends upon the particular power series).

We call the largest $\rho > 0$ such that $\sum_{n=0}^{\infty} a_n x^n$ exists for each $x \in (\rho, \rho)$ the *radius of convergence* of the power series. If $\sum_{n=0}^{\infty} a_n x^n$ diverges for all $x \neq 0$ then we say that the radius of convergence is $\rho = 0$. If $\sum_{n=0}^{\infty} a_n x^n$ is convergent for all $x \in \mathbb{R}$ then we say that the radius of convergence is $\rho = \infty$.

Important: To determine the radius of convergence of a power series it is usually best to use the ratio test.

Exercise: Find the radius of convergence for each of the following power series;

$$\text{(a)} \sum_{n=0}^{\infty} n! x^n; \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{x^n}{n!}; \qquad \text{(c)} \sum_{n=0}^{\infty} x^n.$$

Theorem: (Abel's Limit Theorem) *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series in x with a radius of convergence $\rho > 0$. If we define $f : (-\rho, \rho) \to \mathbb{R}$ by, $f(x) := \sum_{n=0}^{\infty} a_n x^n$ then we have that f is a well-defined continuous function on $(-\rho, \rho)$.*

Exercises

1. Show that $1 < \sum\limits_{n=1}^{\infty} 1/n^2 \leqslant 1 + \sum\limits_{n=2}^{\infty} 1/n(n-1) = 2$.

2. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n(\sqrt{n+1})}$. *Hint:* try to transform this into a telescoping series.

3. Determine whether the series $\sum\limits_{n=1}^{\infty} \sin^2(1/n)$ converges or diverges.

4. Determine whether the series $\sum\limits_{n=1}^{\infty} n^2/(2n^2+1)$ converges or diverges.

5. Use the integral test to determine whether the following series converge or not.

(a) $\sum\limits_{n=1}^{\infty} \log_e(n)/n^2$; (b) $\sum\limits_{n=2}^{\infty} 1/[n \log_e(n)]$; (c) $\sum\limits_{n=2}^{\infty} 1/[n \log_e^2(n)]$.

6. Use induction to show that for each $n \in \mathbb{N}$ and $x > 0$, $1 + nx \leqslant (1+x)^n$ and use your answer to deduce that $2n^n \leqslant (n+1)^n$ for all $n \in \mathbb{N}$. *Hint:* $(n+1)^n = n^n \cdot (1+1/n)^n$.

- (a) Apply the ratio test to show that the series $\sum\limits_{n=1}^{\infty} n!/n^n$ is convergent;
- (b) Deduce from part (a) that $\lim\limits_{n\to\infty} n!/n^n = 0$.

7. Establish whether the series $\sum\limits_{n=1}^{\infty} (n!)^2/(2n)!$ is convergent.

8. Show that the series $\sum\limits_{n=1}^{\infty} (-1)^n/\sqrt{n}$ is convergent.

9. Give an example of two convergent series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ such that the series, $\sum\limits_{n=1}^{\infty} (a_n \cdot b_n)$ is divergent.

10. Let $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be absolutely convergent series.

- (a) Show that the series $\sum\limits_{n=1}^{\infty} (a_n \cdot b_n)$ is convergent. *Hint:* use the fact that the sequence $(b_n : n \in \mathbb{N})$ is bounded and apply the Cauchy condition to deduce the convergence;
- (b) Show that both of the series $\sum\limits_{n=1}^{\infty} a_n^2$ and $\sum\limits_{n=1}^{\infty} b_n^2$ are convergent. *Hint: use the limit comparison test*;
- (c) Show that $\displaystyle\sum_{n=1}^{\infty}(a_n \cdot b_n) \leqslant \sqrt{\sum_{n=1}^{\infty} a_n^2} \cdot \sqrt{\sum_{n=1}^{\infty} b_n^2}$.

11. Find the radius of convergence for each of the following power series;

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n$; (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2 \cdot 4 \cdot 6 \cdot \ \cdot \ \cdot (2n+2)}{1 \cdot 4 \cdot 7 \cdot \ \cdot \ \cdot (3n+1)}\right) x^n$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} x^n$.

12. (Double Summation Theorem) Suppose that $a_{(m,n)} \in [0,\infty)$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$.

(a) Prove by induction (on m_0) that, $\displaystyle\sum_{m=1}^{m_0}\left(\sum_{n=1}^{\infty} a_{(m,n)}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{m_0} a_{(m,n)}\right)$.

(b) Let $\varepsilon > 0$. Show that for some $m_0 \in \mathbb{N}$, $\displaystyle\sum_{m=1}^{m_0}\left(\sum_{n=1}^{\infty} a_{(m,n)}\right) > \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{(m,n)}\right) - \varepsilon$

(c) Let $\varepsilon > 0$. Show that,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{(m,n)} \geqslant \sum_{n=1}^{\infty}\sum_{m=1}^{m_0} a_{(m,n)} = \sum_{m=1}^{m_0}\sum_{n=1}^{\infty} a_{(m,n)} > \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{(m,n)} - \varepsilon$$

(d) Deduce that $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{(m,n)} \geqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{(m,n)}$.

(e) By repeating the steps (a), (b), (c) and (d) with the order of summation interchanged show that $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{(m,n)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{(m,n)}$

Limits of functions

Introduction

In this section of the course we will be concerned with functions that map from either a proper subset of $\mathbb{R}$, or all of $\mathbb{R}$, into $\mathbb{R}$. In particular we will be interested in the case when the domain of our function is an open subset of $\mathbb{R}$.

We will say that a subset A of $\mathbb{R}$ is an *open* subset of $\mathbb{R}$ if for each $x \in A$ there exists a positive real number δ such that $(x-\delta, x+\delta) \subseteq A$. Of course the positive real number δ depends upon the point $x \in A$.

Exercise: Let $a < b$. Show that the set $A := (a, b)$ is an open subset of $\mathbb{R}$, but that the set $[a, b]$ is **not** an open subset of $\mathbb{R}$, ie: open intervals of $\mathbb{R}$ are open sets, but closed intervals are not.

Theorem: (Representation Theorem) *A subset of $\mathbb{R}$ is an open subset of $\mathbb{R}$ if, and only if, it can be expressed as a disjoint union of countably many open intervals.*

Let $A \subseteq \mathbb{R}$ be a non-empty set. A point $x_0 \in \mathbb{R}$ is a *cluster-point* of A if for every $\delta > 0$, $(x_0 - \delta, x_0 + \delta) \cap A \backslash \{x_0\} \neq \emptyset$.

Theorem: *A real number x_0 is a cluster-point of a subset A of $\mathbb{R}$ if, and only if, there is a sequence $(a_n : n \in \mathbb{N})$ in $A \backslash \{x_0\}$ such that $\lim_{n \to \infty} a_n = x_0$.*

Exercise: Let $a < b$. Show that the points a and b are cluster-points of the set (a, b).

We are now in a position where we can rigorously define the limit of a real-valued function defined on an open subset of $\mathbb{R}$.

Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A. A real number L is said to be the *limit of f at x_0* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \delta$. If L is a limit of f at x_0 then we write: $\lim_{x \to x_0} f(x) = L$.

Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A. We say that the *limit of f at x_0 exists* if there exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \delta$.

So loosely speaking a real number L is a limit of f at x_0 if the function f gets closer and closer to L as x gets closer and closer to x_0.

Example: Prove, using an ε - δ argument, that $\lim_{x \to x_0} f(x) = x_0$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by, $f(x) := x$.

Answer: Let ε be any positive real number. If we define $\delta := \varepsilon > 0$ then,
$|f(x) - x_0| = |x - x_0| < \delta = \varepsilon \quad$ for all $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$. $\quad\square$

Exercise: Let x_0 be any real number. Show that for each $\delta > 0$, $|x + x_0| < 2|x_0| + \delta$ for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$. *Hint:* $(x + x_0) = 2x_0 + (x - x_0)$.

Example: Prove, using an ε - δ argument, that $\lim\limits_{x \to x_0} f(x) = x_0^2$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by, $f(x) := x^2$.

Answer: Let ε be any positive real number and let $\delta := \min\{1, \varepsilon/(2|x_0| + 1)\} > 0$ then,

$$|f(x) - x_0^2| = |x^2 - x_0^2| = |x + x_0||x - x_0| \leqslant (2|x_0| + 1)|x - x_0| < (2|x_0| + 1) \cdot \delta < \varepsilon$$

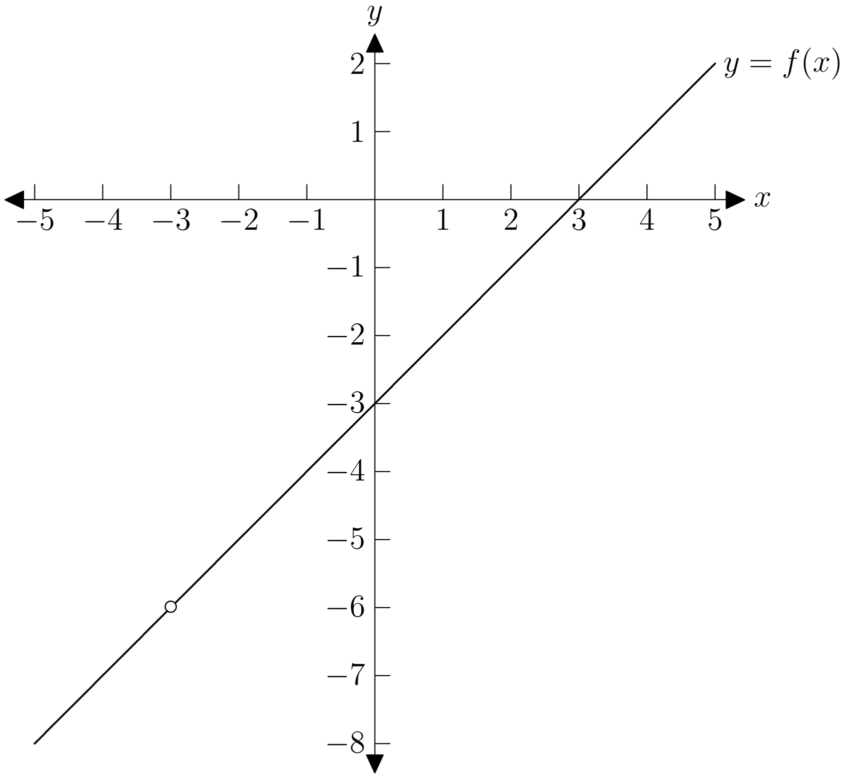
for all $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$. $\quad\square$

Example: Consider the function $f : \mathbb{R}\backslash\{-3\} \to \mathbb{R}$ defined by,

$$f(x) := \frac{x^2 - 9}{x + 3} = \begin{cases} x - 3 & \text{if } x \neq -3; \\ \text{undefined} & x = -3. \end{cases}$$

Find $\lim\limits_{x \to -3} f(x)$, if it exists.

Answer: From the diagram below it is obvious that $\lim\limits_{x \to -3} f(x)$ should be -6. So let us try to provide a rigorous proof of this.



Let ε be any positive real number and let $\delta := \varepsilon > 0$ then,

$$|f(x) - (-6)| = \left|\frac{(x - 3)(x + 3)}{(x + 3)} - (-6)\right| = |(x - 3) + 6| = |x - (-3)| < \delta = \varepsilon$$

for all $x \in \mathbb{R}$ with $0 < |x - (-3)| < \delta$. $\square$

Example: Use an ε - δ argument to show that the limit of the function $f : (0, \infty) \to \mathbb{R}$ defined by, $f(x) := \sqrt{x}$ exists at every point $x_0 \in (0, \infty)$.

Answer: Fix $x_0 \in (0, \infty)$. We shall prove that $\lim_{x \to x_0} f(x) = \sqrt{x_0}$. To this end, let ε be any positive real number and let $\delta := \sqrt{x_0} \cdot \varepsilon > 0$ then,

$$\begin{array}{rcl} |f(x) - \sqrt{x_0}| & = & |\sqrt{x} - \sqrt{x_0}| \\ & = & \left| \dfrac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}} \right| \\ & = & \dfrac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \dfrac{|x - x_0|}{\sqrt{x_0}} < \varepsilon \end{array}$$

for all $x \in (0, \infty)$ with $0 < |x - x_0| < \delta$. $\square$

Observation: Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$. Let x_0 be a cluster-point of A and let L be a real number. If it is possible to show that "$|f(x) - L| \leqslant K \cdot |x - x_0|^p$" for some positive real number K and natural number p, for all $x \in A$, then $\lim\limits_{x \to x_0} f(x) = L$.

Proof: Let ε be any positive real number. If we set $\delta := \sqrt[p]{\varepsilon/K}$ then,
$|f(x) - L| \leqslant K \cdot |x - x_0|^p < K \cdot \delta^p = \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \delta$. $\square$

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the property that $|f(x) - f(y)| \leqslant K \cdot |x - y|^p$ for some $K > 0$ and $p \in \mathbb{N}$ and all $x, y \in \mathbb{R}$. Show that for each $x_0 \in \mathbb{R}$, $\lim\limits_{x \to x_0} f(x)$ exists.

Theorem: *Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A. If the limit of f at x_0 exists then it is unique, ie: f has at most one limit at x_0.*

Proof: Suppose that L_1 and L_2 are limits of f at x_0. We need to show that $L_1 = L_2$, ie: $|L_1 - L_2| = 0$. Let ε be any positive real number. Then by the fact that L_1 is a limit of f at x_0 there exists a $\delta_1 > 0$ such that $|f(x) - L_1| < \varepsilon/2$ for all $x \in A$ with $0 < |x - x_0| < \delta_1$. Similarly, by the fact that L_2 is a limit of f at x_0 there exists a $\delta_2 > 0$ such that $|f(x) - L_2| < \varepsilon/2$ for all $x \in A$ with $0 < |x - x_0| < \delta_2$. Hence for any $x \in A$ with $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$ we have that,

$$|L_1 - L_2| \leqslant |L_1 - f(x)| + |f(x) - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Now since our choice of $\varepsilon > 0$ was arbitrary it must be the case that $|L_1 - L_2| = 0$. $\square$

Theorem: *Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A, then the limit of f at x_0 exists and equals L if, and only if, for each sequence $(a_n : n \in \mathbb{N})$ in $A \backslash \{x_0\}$ converging to x_0, the sequence $(f(a_n) : n \in \mathbb{N})$ converges to L.*

Proof: Suppose that L is the limit of f at x_0 and suppose that $(a_n : n \in \mathbb{N})$ is a sequence in $A \backslash \{x_0\}$ converging to x_0; we need to show that L is the limit of the sequence $(f(a_n) : n \in \mathbb{N})$. Let ε be any positive real number. Since L is the limit of f at x_0 there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \delta$. On the other hand since $(a_n : n \in \mathbb{N})$ converges to x_0 there exists an $N \in \mathbb{N}$ such that $|a_n - x_0| < \delta$ for all $n > N$. Hence it follows that $|f(a_n) - L| < \varepsilon$ for all $n > N$. This shows that $(f(a_n) : n \in \mathbb{N})$ converges to L. Now let us suppose that for each sequence $(a_n : n \in \mathbb{N})$ in $A \backslash \{x_0\}$ converging to x_0, the sequence $(f(a_n) : n \in \mathbb{N})$ converges to L. From this we need to deduce that L is the limit of f at x_0. So in order to obtain a contradiction let us assume that L is **not** the limit of f at x_0. This means that there is some $\varepsilon_0 > 0$ such that for each $\delta > 0$, there is some $a_\delta \in A$ with $0 < |a_\delta - x_0| < \delta$ such that $|f(a_\delta) - L| \geqslant \varepsilon_0$. In particular, this means that for each $n \in \mathbb{N}$, there is some $a_n \in A$ with $0 < |a_n - x_0| < 1/n$ such that $|f(a_n) - L| \geqslant \varepsilon_0$. Now the sequence $(a_n : n \in \mathbb{N})$ converges to x_0, but the sequence $(f(a_n) : n \in \mathbb{N})$ does not converge to L. Hence our assumption that L is not the limit of f at x_0 must be wrong, ie: L must be the limit of f at x_0. ☐

The previous theorem can be useful for showing that the limit of a function does not exist at a particular point. Indeed, to show that the limit of f does not exist at a point x_0 we need only construct a sequence $(a_n : n \in \mathbb{N})$ in $A \backslash \{x_0\}$ converging to x_0 such that $(f(a_n) : n \in \mathbb{N})$ is not convergent.

Example: Let $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ be defined by, $f(x) := |x|/x$. Show that the limit of f at 0 does **not** exist.

Answer: Consider the sequence $(a_n : n \in \mathbb{N})$ defined by, $a_n := (-1)^n/n$ for all $x \in \mathbb{N}$. Then $(a_n : n \in \mathbb{N})$ converges to 0, but $(f(a_n) : n \in \mathbb{N})$ is not Cauchy and hence not convergent. ☐

Exercise: Let $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ be defined by, $f(x) := \cos(1/x)$. Show that the limit of f at 0 does **not** exist.

The following theorem enables us to calculate some limits without having to resort to using ε's and δ's.

Theorem: *Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A. If $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$ then;*

(i) $\displaystyle\lim_{x \to x_0} (f + g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = L_1 + L_2$;

(ii) $\displaystyle\lim_{x \to x_0} cf(x) = c \cdot \left(\lim_{x \to x_0} f(x) \right) = c \cdot L_1$;

(iii) $\displaystyle\lim_{x \to x_0} (f \cdot g)(x) = \left(\lim_{x \to x_0} f(x) \right) \cdot \left(\lim_{x \to x_0} g(x) \right) = L_1 \cdot L_2$;

(iv) $\displaystyle\lim_{x \to x_0} (f \div g)(x) = \left(\lim_{x \to x_0} f(x) \right) \div \left(\lim_{x \to x_0} g(x) \right) = L_1 \div L_2$, provided $L_2 \neq 0$.

Proof: (i) Given $\varepsilon > 0$ choose $\delta_1 > 0$ and $\delta_2 > 0$ so that $|f(x) - L_1| < \varepsilon/2$ for all $x \in A$ with $0 < |x - x_0| < \delta_1$ and $|g(x) - L_2| < \varepsilon/2$ for all $x \in A$ with $0 < |x - x_0| < \delta_2$.

Then $|(f + g)(x) - (L_1 + L_2)| \leqslant |f(x) - L_1| + |g(x) - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$.

(ii) Given $\varepsilon > 0$ choose $\delta > 0$ such that $|f(x) - L_1| < \varepsilon/(|c| + 1)$ for all $x \in A$ with $0 < |x - x_0| < \delta$. Then, $|cf(x) - cL_1| = |c||f(x) - L_1| < \varepsilon \cdot |c|/(|c| + 1) < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \delta$.

(iii) Given $\varepsilon > 0$, choose $\delta_1 > 0$ and $\delta_2 > 0$ so that $|f(x) - L_1| < \sqrt{\varepsilon}$ for all $x \in A$ with $0 < |x - x_0| < \delta_1$ and $|g(x) - L_2| < \sqrt{\varepsilon}$ for all $x \in A$ with $0 < |x - x_0| < \delta_2$.
Then $|(f(x) - L_1)(g(x) - L_2) - 0| = |f(x) - L_1||g(x) - L_2| < \varepsilon$ for all $x \in A$ with $0 < |x-x_0| < \min\{\delta_1, \delta_2\}$. This proves that $\lim_{x\to x_0}(f(x)-L_1)(g(x)-L_2) = 0$. Combining this with (i) and (ii), we have that

$$\begin{aligned}\lim_{x\to x_0}(f\cdot g(x) - L_1L_2) &= \lim_{x\to x_0}\Big((f(x) - L_1)(g(x) - L_2) + L_1(g(x) - L_2) + L_2(f(x) - L_1)\Big)\\ &= 0 + L_1 \cdot \lim_{x\to x_0}(g(x) - L_2) + L_2 \cdot \lim_{x\to x_0}(f(x) - L_1)\\ &= 0 + L_1 \cdot 0 + L_2 \cdot 0 = 0\end{aligned}$$

and so (iii) is obtained.

(iv) Given $\varepsilon > 0$, choose $\delta > 0$ so that $|g(x) - L_2| < |L_2|/2$ for all $x \in A$ with $0 < |x - x_0| < \delta$. Then $|L_2|/2 < |L_2| - |g(x) - L_2| \leqslant |g(x)|$ for all $x \in A$ with $0 < |x - x_0| < \delta$. Next choose $0 < \delta_1 < \delta$ so that $|g(x) - L_2| < |L_2|^2\varepsilon/2$. Then we have

$$|1/g(x) - 1/L_2| = \frac{|g(x) - L_2|}{|L_2||g(x)|} < \frac{2|g(x) - L_2|}{|L_2|^2} < \varepsilon \quad \text{for all } x \in A \text{ with } 0 < |x - x_0| < \delta_1.$$

This proves that $\lim\limits_{x\to x_0}(1/g(x)) = 1/L_2$. The result now follows from (iii). □

Exercise: Give an alternative proof to part(iii) of the previous theorem by;

(a) showing that if the limit of f at x_0 exists then there exists an $M > 0$ and $\delta > 0$ such that $|f(x)| < M$ for all $x \in A$ with $0 < |x - x_0| < \delta$.

(b) using the inequality,
$|f\cdot g(x)-L_1L_2| \leqslant |f(x)g(x)-f(x)L_2|+|f(x)L_2-L_1L_2| = |f(x)||g(x)-L_2|+|L_2||f(x)-L_1|$

Exercise: Give a proof of the previous theorem by using the sequential characterisation of a limit.

Example: Find the limit, at the point $x = 1$, of the function $f : \mathbb{R}\backslash\{-3\} \to \mathbb{R}$ defined by, $f(x) := (x^2 + x - 1)/(x + 3)$.

Answer: $\displaystyle\lim_{x\to 1}\frac{x^2 + x - 1}{x + 3} = \frac{\lim_{x\to 1}(x^2 + x - 1)}{\lim_{x\to 1}(x + 3)} = \frac{(\lim_{x\to 1} x)^2 + \lim_{x\to 1} x + \lim_{x\to 1}(-1)}{\lim_{x\to 1} x + \lim_{x\to 1} 3} = \frac{1}{4}$. □

Theorem: (Sandwich Theorem) *Let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ and $h : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A. If $f(x) \leqslant g(x) \leqslant h(x)$ for all $x \in A$ and $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x)$ then $\lim\limits_{x\to x_0} g(x)$ exists and $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} h(x)$.*

Proof: Let $L := \lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x)$ and suppose $\varepsilon > 0$ is given. Choose $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \delta_1$ and $|h(x) - L| < \varepsilon$

for all $x \in A$ with $0 < |x - x_0| < \delta_2$. Then $-\varepsilon < f(x) - L \leqslant g(x) - L \leqslant h(x) - L < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$ and so $|g(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$. This completes the proof. $\square$

Example: Find the limit of the function $f : \mathbb{R} \to \mathbb{R}$ defined by, $g(x) := x \cdot \sin(x)$ at the point $x = 0$.

Answer: First note that $|g(x)| \leqslant |x|$ for all $x \in \mathbb{R}$. Then define the functions $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ by, $f(x) := -|x|$ and $h(x) := |x|$. We may now apply the sandwich theorem since, $f(x) \leqslant g(x) \leqslant h(x)$ for all $x \in \mathbb{R}$ and $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} h(x) = 0$, to obtain that $\lim\limits_{x \to 0} g(x) = 0$. $\square$

Theorem: (Cauchy Condition) *Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A. Then $\lim_{x \to x_0} f(x)$ exists if, and only if, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in A$ with $0 < |x - x_0| < \delta$ and $0 < |y - x_0| < \delta$.*

Proof: Suppose that $\lim_{x \to x_0} f(x) = L$. Let ε be any positive real number. Then by the definition of $\lim_{x \to x_0} f(x)$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon/2$ for all $x \in A$ with $0 < |x - x_0| < \delta$. Therefore, $|f(x) - f(y)| \leqslant |f(x) - L| + |L - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $x, y \in A$ with $0 < |x - x_0| < \delta$ and $0 < |y - x_0| < \delta$.

Converse: We will use the sequential definition of a limit to deduce that $\lim_{x \to x_0} f(x)$ exists. To this end let $(a_n : n \in \mathbb{N})$ be any sequence in $A \backslash \{x_0\}$ converging to x_0. It is easy to see from the hypotheses that the sequence $(f(a_n) : n \in \mathbb{N})$ is a Cauchy sequence and hence a convergent sequence. So to complete the proof we must show that all the sequences of the form: $(f(a_n) : n \in \mathbb{N})$ with $(a_n : n \in \mathbb{N})$ in $A \backslash \{x_0\}$ and $\lim_{n \to \infty} a_n = x_0$ have a common limit. To do this, let $(a_n : n \in \mathbb{N})$ and $(b_n : n \in \mathbb{N})$ be any two sequences in $A \backslash \{x_0\}$ converging to x_0 and let $L_1 := \lim_{n \to \infty} f(a_n)$ and $L_2 := \lim_{n \to \infty} f(b_n)$; we must show that $L_1 = L_2$, ie: $|L_1 - L_2| = 0$. Let ε be any positive real number. Then by the fact that L_1 is the limit of $(f(a_n) : n \in \mathbb{N})$ there exists an $N_1 \in \mathbb{N}$ such that $|f(a_n) - L_1| < \varepsilon/3$ for all $n > N_1$. Similarly, by the fact that L_2 is the limit of $(f(b_n) : n \in \mathbb{N})$ there exists an $N_2 \in \mathbb{N}$ such that $|f(b_n) - L_2| < \varepsilon/3$ for all $n > N_2$. Now by our hypotheses and the fact that both sequences $(a_n : n \in \mathbb{N})$ and $(b_n : n \in \mathbb{N})$ converge to x_0 there exists an $N_3 \in \mathbb{N}$ such that $|f(a_n) - f(b_n)| < \varepsilon/3$ for all $n > N_3$. Hence for any $n > \max\{N_1, N_2, N_3\}$,

$$|L_1 - L_2| \leqslant |L_1 - f(a_n)| + |f(a_n) - f(b_n)| + |f(b_n) - L_2| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Now since our choice of $\varepsilon > 0$ was arbitrary it must be the case that $|L_1 - L_2| = 0$. $\square$

Let $f : \mathbb{R} \to \mathbb{R}$. A real number L is said to be the *limit of f as x tends to ∞* if for each $\varepsilon > 0$ there exists an $N > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > N$ and we write: $\lim\limits_{x \to \infty} f(x) = L$.

Let $f : \mathbb{R} \to \mathbb{R}$. We say that the *limit of f as x tends to ∞ exists* if there exists an $L \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists an $N > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > N$.

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$. Show that if $\lim\limits_{x \to \infty} f(x)$ exists then there exists an $M > 0$ and an $N > 0$ such that $|f(x)| < M$ for all $x > N$.

Theorem: *Let* $f : \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$. *If* $\lim\limits_{x\to\infty} f(x) = L_1$ *and* $\lim\limits_{x\to\infty} g(x) = L_2$ *then,*

(i) $\lim\limits_{x\to\infty}(f+g)(x) = \lim\limits_{n\to\infty} f(x) + \lim\limits_{x\to\infty} g(x) = L_1 + L_2$*;*

(ii) $\lim\limits_{x\to\infty} cf(x) = c \cdot \Big(\lim\limits_{x\to\infty} f(x)\Big)$*;*

(iii) $\lim\limits_{x\to\infty}(f \cdot g)(x) = \Big(\lim\limits_{x\to\infty} f(x)\Big) \cdot \Big(\lim\limits_{x\to\infty} g(x)\Big) = L_1 \cdot L_2$*;*

(iv) $\lim\limits_{x\to\infty}(f \div g)(x) = \Big(\lim\limits_{x\to\infty} f(x)\Big) \div \Big(\lim\limits_{x\to\infty} g(x)\Big) = L_1 \div L_2$*, provided* $L_2 \neq 0$*.*

Proof: The proof of this is left as an exercise for the reader. □

Example: Calculate $\lim\limits_{x\to\infty} f(x)$, where f is defined by, $f(x) := (2x^3 - 3x + 6)/(x^3 - 5x + 1)$.

Answer: Now for $x \neq 0$, $f(x) = (2 - 3/x^2 + 6/x^3)/(1 - 5/x^2 + 1/x^3)$ and so,

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\left(\frac{2 - 3/x^2 + 6/x^3}{1 - 5/x^2 + 1/x^3}\right) = \frac{\lim_{x\to\infty}(2 - 3/x^2 + 6/x^3)}{\lim_{x\to\infty}(1 - 5/x^2 + 1/x^3)} \\ &= \frac{(\lim_{x\to\infty} 2) - (\lim_{x\to\infty} 3/x^2) + (\lim_{x\to\infty} 6/x^3)}{(\lim_{x\to\infty} 1) - (\lim_{x\to\infty} 5/x^2) + (\lim_{x\to\infty} 1/x^3)} = \frac{2 - 0 + 0}{1 - 0 + 0} = 2. \quad \square\end{aligned}$$

We say that a function $f : \mathbb{R} \to \mathbb{R}$ *diverges to* ∞ *as* x *tends to* ∞ if for each $M > 0$ there exists an $N > 0$ such that $f(x) > M$ for all $x > N$ and we write: $\lim\limits_{x\to\infty} f(x) = \infty$.

Moreover we say that a function $f : \mathbb{R} \to \mathbb{R}$ *diverges to* $-\infty$ *as* x *tends to* ∞ if for each $M > 0$ there exists an $N > 0$ such that $f(x) < -M$ for all $x > N$ and we write: $\lim\limits_{x\to\infty} f(x) = -\infty$.

Example: Give a formal proof that $\lim\limits_{x\to\infty} f(x) = \infty$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by, $f(x) := (x^5 + 5x - 9)/(x^4 + x^3 + 7)$.

Answer: Let M be any positive real number and let $N := \max\{2, 3M\}$, then

$$f(x) = \frac{x^5 + 5x - 9}{x^4 + x^3 + 7} \geqslant \frac{x^5}{x^4 + x^4 + x^4} = x/3 > N/3 \geqslant M \quad \text{for all } x > N. \quad \square$$

Observation: Let $f : \mathbb{R} \to \mathbb{R}$. If it is possible to show that "$f(x) \geqslant K \cdot x^p$" for some positive real number K and natural number p, for all $x \in \mathbb{R}$, then $\lim\limits_{x\to\infty} f(x) = \infty$.

Proof: Let M be any positive real number. If we set $N := \sqrt[p]{M/K}$ then we have that $f(x) \geqslant K \cdot x^p > K \cdot N^p = M$ for all $n > N$. □

Exercise: Give a formal proof that $\lim\limits_{x\to\infty} f(x) = \infty$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by, $f(x) := (\sqrt{x^4 + x^2 + 1})/x$.

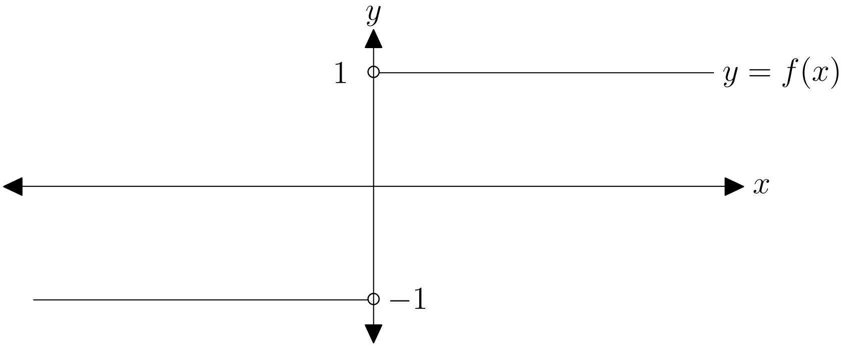
Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let x_0 be a cluster-point of A. We say that f *diverges* to ∞ at x_0 if for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ for all $x \in A$ with $0 < |x - x_0| < \delta$ and we write: $\lim\limits_{x\to x_0} f(x) = \infty$.

Likewise we say that f diverges to $-\infty$ at x_0 if for each $M > 0$ there exists a $\delta > 0$ such that $f(x) < -M$ for all $x \in A$ with $0 < |x - x_0| < \delta$ and we write: $\lim\limits_{x \to x_0} f(x) = -\infty$.

One-sided limits

Consider the function $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ defined by,

$$f(x) := \frac{|x|}{x} = \begin{cases} 1 & x > 0; \\ -1 & x < 0. \end{cases}$$



Clearly $\lim_{x \to 0} f(x)$ does not exist (why?). However the limit "half exists". That is to say that both of the "one-sided" limits exist.

Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$. If x_0 is a cluster-point of $A \cap (x_0, \infty)$ then we say that a real number L_1 is the *right-hand limit* of f at x_0 if for each $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that $|f(x) - L_1| < \varepsilon$ for all $x \in A \cap (x_0, x_0 + \delta_1)$ and we write: $\lim\limits_{x \to x_0^+} f(x) = L_1$.

If x_0 is a cluster-point of $A \cap (-\infty, x_0)$ then we say that a real number L_2 is the *left-hand limit* of f at x_0 if for each $\varepsilon > 0$ there exists a $\delta_2 > 0$ such that $|f(x) - L_2| < \varepsilon$ for all $x \in A \cap (x_0 - \delta_2, x_0)$ and we write: $\lim\limits_{x \to x_0^-} f(x) = L_2$.

So in the previous example $\lim\limits_{x \to 0^-} f(x) = -1$ and $\lim\limits_{x \to 0^+} f(x) = 1$.

Theorem: *Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let $x_0 \in A$. Then $\lim\limits_{x \to x_0^-} f(x) = L$ and $\lim\limits_{x \to x_0^+} f(x) = L$ if, and only if, $\lim\limits_{x \to x_0} f(x) = L$.*

Proof: Suppose that $\lim_{x \to x_0} f(x) = L$. Let $\varepsilon > 0$ be any positive real number. Then by the definition of $\lim_{x \to x_0} f(x) = L$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \varepsilon$. Hence, $|f(x) - L| < \varepsilon$ for all $x \in A \cap (x_0 - \delta, x_0)$ and $|f(x) - L| < \varepsilon$ for all $x \in A \cap (x_0, x_0 + \delta)$. This shows that if $\lim_{x \to x_0} f(x) = L$ then $\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0^+} f(x) = L$. Conversely, suppose $\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0^+} f(x) = L$. Let ε be any positive real number. Then by the definitions of $\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0 +} f(x) = L$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A \cap (x_0 - \delta_1, x_0)$ and $|f(x) - L| < \varepsilon$ for all $x \in A \cap (x_0, x_0 + \delta_2)$. Therefore, $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \min\{\delta_1, \delta_2\}$. Hence $\lim\limits_{x \to x_0} f(x) = L$. $\square$

Example: Let $f : \mathbb{R}\backslash\{0,-1\} \to \mathbb{R}$ be defined by, $f(x) := x/(|x|(x+1))$. Calculate $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

Answer: For $x > 0$, $f(x) = 1/(x+1)$ and so the right-hand limit of f at 0 is: 1. For $x < 0$, $f(x) = (-1)/(x+1)$ and so the left-hand limit of f at 0 is: -1. $\square$

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by, $f(x) := |x| \cdot x$. Show that $\lim_{x\to 0} f(x) = 0$. *Hint:* calculate the left-hand and right-hand limits separately.

Exercises

1. Prove, using an ε - δ argument, that $\lim_{x\to 2} \dfrac{x^3-8}{x-2} = 12$.

2. Use both the ε - δ and the sequential formulations of the notion of a limit to establish the following; (a) $\lim_{x\to 2} \dfrac{1}{1-x} = -1$; (b) $\lim_{x\to 1} \dfrac{x}{1+x} = \dfrac{1}{2}$; (c) $\lim_{x\to 0} \dfrac{x^2}{|x|} = 0$; (d) $\lim_{x\to 1} \dfrac{x^2-x+1}{x+1} = \dfrac{1}{2}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational and $f(x) := 0$ if x is irrational. Show that f has a limit at $x = 0$. Use a sequential argument to show that if $x \neq 0$ then f does not have a limit at x.

4. Let $f : \mathbb{R} \to \mathbb{R}$. Show that if $\lim_{x\to x_0} |f(x)| = 0$ then $\lim_{x\to x_0} f(x) = 0$.

5. Calculate: $\lim_{x\to 2} \dfrac{x^2-4}{3x-6}$.

6. Evaluate the limit: $\lim_{x\to 0} \dfrac{\sqrt[3]{x+8}-2}{x}$. *Hint:* write $y^3 := x + 8$.

7. Evaluate the limit: $\lim_{x\to 0} \dfrac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$.

8. Use the sandwich theorem to show that $\lim_{x\to 0} x^2 \cdot (1+x)^{100} = 0$.

9. Prove by induction that, $-|x| \leqslant x^n \leqslant |x|$, for all $x \in [-1,1]$ and all $n \in \mathbb{N}$. Use this to show that $\lim_{x\to 0} x^n = 0$.

10. Calculate: $\lim_{x\to\infty} \dfrac{2x^3-3x+1}{x^3+3x^2+2}$ and $\lim_{x\to\infty} \left(\dfrac{2x^4+3x+1}{x^5+4x^2+3} + \dfrac{x^2+\sqrt{x}}{3x^2+5x+3}\right)$.

11. Prove, using an ε - N argument, that $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x} = 1$.

12. Prove, using an M - δ argument, that $\lim_{x\to 0} \dfrac{\sqrt{x^2+1}}{x^2} = \infty$.

13. Calculate the left-hand and right-hand limits of the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by, $f(x) := (\sqrt{x^2+1})/x$, at the point $x = 0$.

14. Let $f : \mathbb{R} \to \mathbb{Z}$ be defined by, $f(x) :=$"the integer part of x". Show that the left-hand and right-hand limits of f exist everywhere. Does $\lim_{x\to x_0} f(x)$ exist if $x_0 \in \mathbb{Z}$?

Continuity of functions

Introduction

Perhaps the most important class of functions in the whole of mathematics is the class of continuous functions. Intuitively these are those functions that can be drawn on a piece of paper without lifting your pen off the paper, ie: the graphs of these functions appear as a "single piece".

Unfortunately the above description cannot serve as a definition. The first reason for this is that there are continuous functions such as: $f(x) := x \cdot \sin(1/x)$ for $x \neq 0$ and $f(x) := 0$ for $x = 0$ that cannot be drawn. In fact it is not possible to draw the graph of any function, not even $f(x) := x$. One can, at best, only approximately depict the graph of a function. However there is an even deeper reason why the above description cannot serve as a definition, namely, the functions that we are interested in map from $\mathbb{R}$ into $\mathbb{R}$; which is a mathematical construct, built up from set theory. The *real line* is **not** an object of "reality" but only an object of abstraction. But even forgetting the above "technical" argument, from a completely pragmatic point of view, the above definition is not very useful. For example, how would one go about showing that the sum of two continuous functions is again continuous? Or for that matter, how would one go about deriving any of the properties that a continuous function may or may not have?

With this preliminary discussion out of the way, let us now give the formal "mathematical" definition of continuity which hopefully captures, at the intuitive level, the notion of continuity described above.

Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let $x_0 \in A$. We say that f is *continuous* at x_0 if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in A$ with $|x - x_0| < \delta$. Note: this is equivalent to saying that $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Theorem: *Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let $x_0 \in A$. Then f is continuous at x_0 in A if, and only if, for every sequence $(a_n : n \in \mathbb{N})$ in A converging to x_0, the sequence $(f(a_n) : n \in \mathbb{N})$ converges to $f(x_0)$.*

Proof: This follows directly from the sequential characterisation of a limit. $\square$

A function $f : A \to \mathbb{R}$ is said to be *continuous from the right* at $x_0 \in A$ if, $\lim_{x \to x_0^+} f(x) = f(x_0)$ and is said to be *continuous from the left* at $x_0 \in A$ if, $\lim_{x \to x_0^-} f(x) = f(x_0)$.

Theorem: *Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let $x_0 \in A$. Then f is continuous at x_0 if, and only if, f is continuous from the left and from the right.*

Proof: The proof of this follows from our earlier work on one-sided limits $\square$

Let A be a non-empty open subset of $\mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is *continuous on* A (or simply *continuous*) if f is continuous at every point of A. A function $f : [a, b] \to \mathbb{R}$ is said to be *continuous on* $[a, b]$ (or simply *continuous*) if it is continuous on (a, b), continuous from the right at a and continuous from the left at b.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by, $f(x) := x^4$. Show that f is continuous on $\mathbb{R}$.

Answer: Let $x_0 \in \mathbb{R}$, then $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} x^4 = (\lim\limits_{x \to x_0} x)^4 = x_0^4 = f(x_0)$. $\square$

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by, $f(x) := 0$ if x is rational and $f(x) := 1$ if x is irrational. Show that f is **not** continuous at any point of $\mathbb{R}$.

Example: Let $f : (0, \infty) \to \mathbb{R}$ be defined in the following way. If x is irrational then $f(x) := 0$ and if x is rational then we may write x in the form: m/n with $m, n \in \mathbb{N}$ and g.c.d.$(m, n) = 1$ and define $f(x) := 1/n$. Show that f is continuous at x_0 if, and only if, x_0 is irrational.

Answer: If x_0 is a rational number then we may choose a sequence $(a_n : n \in \mathbb{N})$ of irrational numbers converging to x_0. Then $0 = \lim_{n\to\infty} f(a_n) \neq f(x_0) > 0$. Hence f is not continuous at $x_0 \in \mathbb{Q}$. On the other hand, if x_0 is irrational and $\varepsilon > 0$ is given then there exists an $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$. Now there are only finitely many rational numbers with denominator less than n_0 in the interval $(x_0 - 1, x_0 + 1)$. Hence we can choose $\delta > 0$ small enough so that the interval $(x_0 - \delta, x_0 + \delta)$ contains no rational numbers with denominator less than n_0. Therefore, $|f(x) - f(x_0)| = |f(x)| \leqslant 1/n_0 < \varepsilon$ for all $x \in (0, \infty)$ with $|x - x_0| < \delta$. Thus f is continuous at the irrational number x_0. $\square$

Exercise: Use an ε - δ argument to show that the function $f : (0, \infty) \to \mathbb{R}$ defined by, $f(x) := \sqrt{x}$ is continuous on $(0, \infty)$.

Theorem: *Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let $x_0 \in A$. If both f and g are continuous at x_0 then;*

(i) $f + g$ is continuous at x_0;

(ii) $c \cdot f$ is continuous at x_0;

(iii) $f \cdot g$ is continuous at x_0;

(iv) $f \div g$ is continuous at x_0, provided $g(x_0) \neq 0$.

Proof: The proof of this follows from our earlier results on limits. $\square$

Corollary: Let p and q be polynomials. Then the function $r : A \to \mathbb{R}$ defined by, $r(x) := p(x)/q(x)$ is continuous on A, where $A := \{x \in \mathbb{R} : q(x) \neq 0\}$.

Theorem: *Let $f : A \to B$ and $g : B \to \mathbb{R}$ be defined on non-empty open subsets A and B of $\mathbb{R}$. If f is continuous at $x_0 \in A$ and g is continuous at $f(x_0)$ then $g \circ f$ is continuous at x_0.*

Proof: Let ε be any positive real number. Now since g is continuous at $f(x_0)$ there exists a $\delta' > 0$ such that $|g(y) - g(f(x_0))| < \varepsilon$ for all $y \in B$ with $|y - f(x_0)| < \delta'$. Next, we set $\varepsilon' := \delta'$ and obtain from the continuity of f at x_0 a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon' = \delta'$ for all $x \in A$ with $|x - x_0| < \delta$. Hence, $|(g \circ f)(x) - (g \circ f)(x_0)| = |g(f(x)) - g(f(x_0))| < \varepsilon$ for all $x \in A$ with $|x - x_0| < \delta$. $\square$

Some theorems on continuous functions

Exercise: Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$ and let $x_0 \in A$. Show that if f is continuous at x_0 then there exists an $M > 0$ and $\delta > 0$ so that $|f(x)| \leqslant M$ for all $x \in A$ with $|x - x_0| < \delta$.

A function $f : [a,b] \to \mathbb{R}$ is said to be *bounded* on $[a,b]$ if there exists an $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in [a,b]$.

Theorem: (Boundedness Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Then f is bounded on $[a,b]$.*

Proof: Let $\mathcal{S} := \{x \in [a,b] : f \text{ is bounded on } [a,x]\}$. The set $\mathcal{S}$ is non-empty since $a \in \mathcal{S}$. Also $\mathcal{S}$ is bounded above by b. Therefore $s := \sup \mathcal{S}$ exists. We claim that $s = b$. Indeed, if $s < b$ then we may choose $0 < \delta < \min\{s-a, b-s\}$ so that f is bounded on $(s-\delta, s+\delta)$. Now f is bounded on $[a, s-\delta]$ and so f is bounded on $[a, s+\delta/2] \subseteq [a, s-\delta] \cup (s-\delta, s+\delta)$; which contradicts that s is an upper bound for $\mathcal{S}$. Hence $s = b$. Now f is continuous from the left at b and so there exists a $0 < \delta$ such that f is bounded on $(b-\delta, b]$. However f is also bounded on $[a, b-\delta]$ and so bounded on $[a, b-\delta] \cup (b-\delta, b] = [a,b]$. $\square$

Theorem: (Min-Max Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Then there exist points $x_1, x_2 \in [a,b]$ such that $f(x_1) = \sup\{f(x) : x \in [a,b]\}$ and $f(x_2) = \inf\{f(x) : x \in [a,b]\}$.*

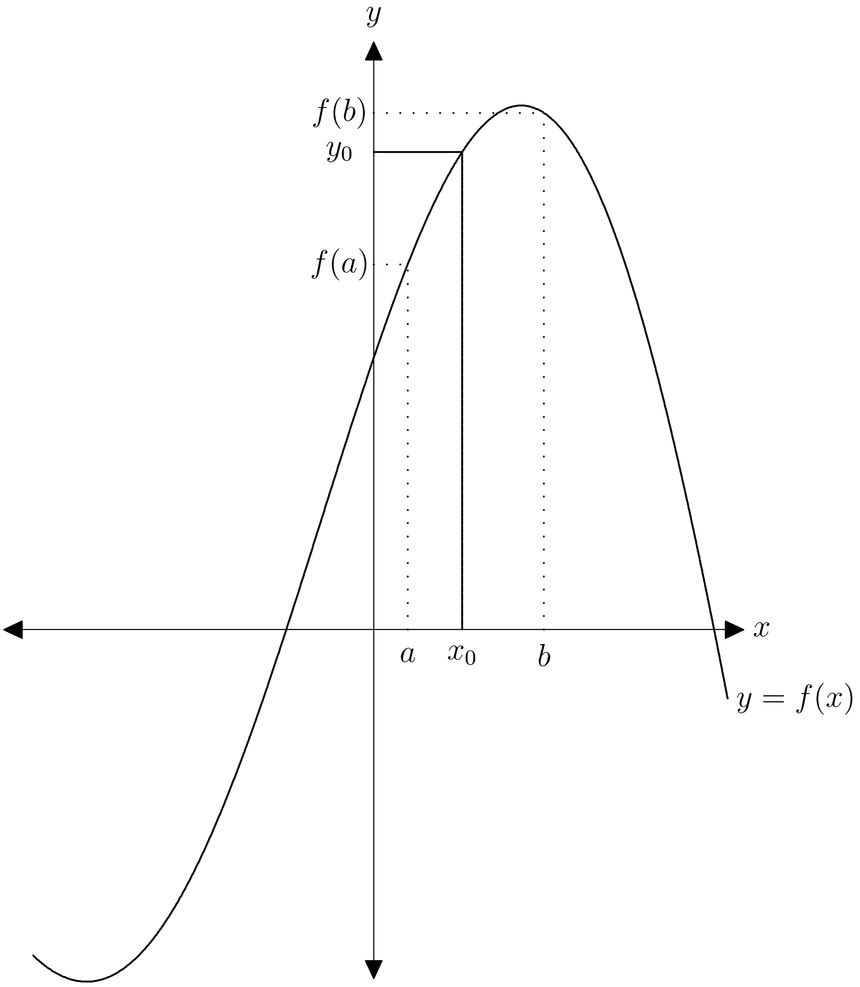
Proof: We shall only prove that there exists an $x_1 \in [a,b]$ such that $f(x_1) = \sup\{f(x) : x \in [a,b]\} := M$. Let us assume, in order to obtain a contradiction, that no such point $x_1 \in [a,b]$ exists. Then $f(x) < M$ for all $x \in [a,b]$ and so the function $g : [a,b] \to \mathbb{R}$ defined by, $g(x) := 1/(M - f(x))$ is continuous. Now by the previous theorem there exists a $K > 0$ such that $g(x) \leqslant K$ for all $x \in [a,b]$, ie: $1/K \leqslant M - f(x)$ or, $f(x) \leqslant M - (1/K)$ for all $x \in [a,b]$. Hence $M - (1/K) < M$ is an upper bound for $\{f(x) : x \in [a,b]\}$; which is impossible since M was defined to have been the "least upper bound" for $\{f(x) : x \in [a,b]\}$. Hence there must be some point $x_1 \in [a,b]$ such that $f(x_1) = M$. $\square$

Theorem: (Intermediate Value Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If $f(a)$ and $f(b)$ have opposite signs then there exists a point x_0 in (a,b) such that $f(x_0) = 0$.*

Proof: Suppose that $f(a) < 0 < f(b)$. Let $\mathcal{S} := \{x \in [a,b] : f(x) < 0\}$. The set $\mathcal{S}$ is non-empty since $a \in \mathcal{S}$. Also $\mathcal{S}$ is bounded above by b. Therefore, $s := \sup \mathcal{S} \leqslant b$ exists. We claim that $f(s) := 0$. To see this, note that (i) if $f(s) < 0$ then there exists a $\delta > 0$ such that $f(y) < 0$ for all $y \in (s-\delta, s+\delta)$; (ii) if $f(s) > 0$ then there exists a $\delta > 0$ such that $f(y) > 0$ for all $y \in (s-\delta, s+\delta)$. So in either case we obtain a contradiction. Therefore, it must be the case that $f(s) = 0$. The proof for the case when $f(b) < 0 < f(a)$ is similar. $\square$

Theorem: (Bolzano Intermediate Value Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If $f(a) < y_0 < f(b)$ (or vice versa) then there exists a point*

$x_0 \in (a, b)$ such that $f(x_0) = y_0$.



Proof: Consider the function $g : [a, b] \to \mathbb{R}$ defined by, $g(x) := f(x) - y_0$. Then $g(a)$ and $g(b)$ have opposite signs and so there exists a point $x_0 \in (a, b)$ such that $g(x_0) = f(x_0) - y_0 = 0$. Therefore, $f(x_0) = y_0$. $\square$

Example: Show that there exists an $x_0 \in (0, 1)$ such that $\cos(x_0) = x_0$.

Answer: Let $f : [0, 1] \to \mathbb{R}$ be defined by, $f(x) := \cos(x) - x$. Then f is a continuous function on $[0, 1]$ and $f(1) < 0 < f(0)$. Therefore, by the intermediate value theorem there exists a point $x_0 \in (0, 1)$ such that $f(x_0) = 0$, ie: $\cos(x_0) = x_0$. $\square$

Exercise: Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. Show that $f([a, b])$ is a closed interval of $\mathbb{R}$. *Hint:* show that $f([a, b]) = [m, M]$, where $m := \inf\{f(x) : x \in [a, b]\}$ and $M := \sup\{f(x) : x \in [a, b]\}$.

Monotone functions

Let $f : A \to \mathbb{R}$ be defined on a non-empty open subset A of $\mathbb{R}$. Then we say that f is *increasing* (*strictly increasing*) if $f(x_1) \leqslant f(x_2)$ ($f(x_1) < f(x_2)$) whenever $x_1 < x_2$.

Similarly, we say that f is *decreasing* (*strictly decreasing*) if $f(x_2) \leqslant f(x_1)$ $(f(x_2) < f(x_1))$ whenever $x_1 < x_2$.

Theorem: *Suppose that* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is increasing on* $[a, b]$. *Let* $x_0 \in (a, b)$ *then* $\lim\limits_{x \to x_0^-} f(x)$ *and* $\lim\limits_{x \to x_0^+} f(x)$ *exist and* $\lim\limits_{x \to x_0^-} f(x) \leqslant f(x_0) \leqslant \lim\limits_{x \to x_0^+} f(x)$.

Proof: We shall show that $\lim_{x \to x_0^-} f(x) \leqslant f(x_0)$. Let $\mathcal{S} := \{f(x) : a \leqslant x < x_0\}$. Then $\mathcal{S}$ is non-empty (since $f(a) \in \mathcal{S}$) and bounded above by $f(x_0)$. Hence $s := \sup \mathcal{S} \leqslant f(x_0)$ exists. We claim that $\lim_{x \to x_0^-} f(x) = s$. To justify this, let us consider ε, an arbitrary positive real number. Now $s - \varepsilon$, is **not** an upper bound for the set $\mathcal{S}$ as s is the least upper bound for $\mathcal{S}$. Hence there exists an element $x_\varepsilon \in [a, x_0)$ such that $s - \varepsilon < f(x_\varepsilon)$. Set $\delta := x_0 - x_\varepsilon > 0$ then $s - \varepsilon < f(x_\varepsilon) \leqslant f(x) \leqslant s < s + \varepsilon$ for all $x \in (x_0 - \delta, x_0)$, ie: $-\varepsilon < f(x) - s < \varepsilon$ for all $x \in (x_0 - \delta, x_0)$ and so $|f(x) - s| < \varepsilon$ for all $x \in (x_0 - \delta, x_0)$. The proof that $f(x_0) \leqslant \lim\limits_{x \to x_0^+} f(x)$ is similar. □

Corollary: *Suppose that* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is increasing on* $[a, b]$. *Let* $x_0 \in (a, b)$ *then* f *is continuous at* x_0 *if, and only if,* $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x)$.

Exercise: Let $f : [a, b] \to [c, d]$ be 1-to-1 and onto. Show that the inverse of f, denoted f^{-1} exists and maps $[c, d]$ onto $[a, b]$.

Exercise: Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is increasing on $[a, b]$. Show that f is continuous if, and only if, $f([a, b]) = [f(a), f(b)]$.

Theorem: (Continuous Inverse Theorem) *Suppose that* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is strictly increasing and continuous on* $[a, b]$. *Then the inverse mapping* f^{-1} *exists and is strictly increasing and continuous on* $[f(a), f(b)]$.

Proof: First, by the intermediate value theorem f maps $[a, b]$ onto $[f(a), f(b)]$ and since f is strictly increasing f is 1-to-1. Hence f^{-1} exists. Moreover, f^{-1} maps from $[f(a), f(b)]$ onto $[a, b]$. Hence by the previous exercise, to show that f^{-1} is strictly increasing and continuous we need only show that f^{-1} is strictly increasing. To this end, let y_1 and $y_2 \in [f(a), f(b)]$ with $y_1 < y_2$. Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in [a, b]$. Now $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$ for if $x_2 \leqslant x_1$ then $y_2 = f(x_2) \leqslant f(x_1) = y_1$; which contradicts the assumption that $y_1 < y_2$. □

Exercises

1. Let $f : (0, \infty) \to \mathbb{R}$ be defined in the following way. If x is irrational then $f(x) := 0$ and if x is rational then we may write x in the form: m/n with $m, n \in \mathbb{N}$ and g.c.d.$(m, n) = 1$ and define $f(x) := n$. Show that f is nowhere continuous on $(0, \infty)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that if $f = g$ on $\mathbb{Q}$ then $f = g$, ie: $f(x) = g(x)$ for all $x \in \mathbb{R}$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and satisfies the algebraic relation: $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = ax$ for some $a \in \mathbb{R}$ and all $x \in \mathbb{R}$. *Hint:* show that $f(x) = f(1) \cdot x$ on $\mathbb{Q}$ and then use the result from **2.**

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and satisfies the algebraic relation: $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = a^x$ for some $a \in \mathbb{R}$ and all $x \in \mathbb{R}$. *Hint:* show that $f(x) = f(1)^x$ on $\mathbb{Q}$ and use the result from **2.** as well as the fact that $x \mapsto a^x$ is continuous on $\mathbb{R}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $(a_n : n \in \mathbb{N})$ be the sequence defined by, $a_{n+1} := f(a_n)$ for all $n \in \mathbb{N}$ and $a_1 \in \mathbb{R}$. Show that if the sequence $(a_n : n \in \mathbb{N})$ converges to some point a_∞ then $a_\infty = f(a_\infty)$. *Hint:* justify the equation: $f(\lim\limits_{n\to\infty} a_n) = \lim\limits_{n\to\infty} f(a_n)$.

6. Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function. Show that if $f(0) = f(2\pi)$ then there exists a point $x_0 \in [0, \pi]$ such that $f(x_0) = f(x_0 + \pi)$. *Hint:* consider the function $g : [0, \pi] \to \mathbb{R}$ defined by, $g(x) := f(x) - f(x + \pi)$.

7. Let $f : [a, b] \to [a, b]$ be continuous function. Show that there exists a point $x_0 \in [a, b]$ such that $f(x_0) = x_0$. *Hint:* it may be constructive to consider the function $g : [a, b] \to \mathbb{R}$ defined by, $g(x) := f(x) - x$.

8. Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous. If $f(x) > 0$ for all $x \in [a, b]$ show that there exists an $\varepsilon > 0$ such that $f(x) \geqslant \varepsilon$ for all $x \in [a, b]$.

9. Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous. If for each $x \in [a, b]$ there exists a point $y \in [a, b]$ such that $|f(y)| \leqslant 1/2|f(x)|$. Show that there exists a point $x_0 \in [a, b]$ such that $f(x_0) = 0$.

10. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous functions. Show that if for each $\varepsilon > 0$ there exists a point $x_\varepsilon \in [a, b]$ such that $|f(x_\varepsilon) - g(x_\varepsilon)| < \varepsilon$ then there exists a point $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by, $f(x) := x^3$. Show that the inverse of f exists and is strictly increasing and continuous on $\mathbb{R}$. *Hint:* it may be helpful to use use the inequality, $x^2 + xy + y^2 \geqslant (x^2 - 2|xy| + y^2) + |xy| = (|x| - |y|)^2 + |x||y|$.

12. Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is strictly decreasing and continuous on $[a, b]$. Show that the inverse mapping f^{-1} exists and is strictly decreasing and continuous on $[f(b), f(a)]$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Show that the set D of all point where f is discontinuous is countable. *Hint:* since f is increasing f is discontinuous at a point $x_0 \in \mathbb{R}$ if, and only if, $f^-(x_0) := \lim_{x\to x_0^-} f(x) < \lim_{x\to x_0^+} f(x) := f^+(x_0)$. Now define the mapping $r : D \to \mathbb{Q}$ in the following way. For each $d \in D$ choose a rational number $r_d \in (f^-(x_0), f^+(x))$ and define $r(d) := r_d$. Then show that r is 1-to-1.

14. Let $\{r_n : n \in \mathbb{N}\}$ be an enumeration of the rational numbers and define $f : \mathbb{R} \to \mathbb{R}$ by, $f(x) := \sum_{r_n < x} 1/2^n$, ie: the sum is taken over all $n \in \mathbb{N}$ such that $r_n < x$.

(a) Show that f is strictly increasing on $\mathbb{R}$.

(b) Show that f is discontinuous at each point of $\mathbb{Q}$.

(c) Show that f is continuous at each irrational number.

Differentiation

Introduction

Suppose that $a < b$ and $f : (a, b) \to \mathbb{R}$. We say that a real number L is the *derivative* of f at a point $x_0 \in (a, b)$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon \quad \text{for all } x \in (a, b) \text{ with } 0 < |x - x_0| < \delta.$$

In this case we say that f is *differentiable* at x_0 and we write $f'(x_0)$ for L. An equivalent way of phrasing this is: the derivative of f at x_0 is $f'(x_0) := \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by, $f(x) := x^2$. Find the derivative of f at x_0.

Answer: We claim that $f'(x_0) = 2x_0$. Let ε be an arbitrary positive real number and set $\delta := \varepsilon$ then,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - 2x_0 \right| = \left| \frac{(x + x_0)(x - x_0)}{x - x_0} - 2x_0 \right| = |x - x_0| < \varepsilon$$

for all $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$. Alternatively we could just use the limit laws to get that $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim\limits_{x \to x_0} (x + x_0) = 2x_0$. $\square$

Theorem: *Suppose that $a < b$ and $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x_0 \in (a, b)$. Then f is continuous at x_0.*

Proof: Let $x_0 \in (a, b)$. Then,

$$f(x) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \quad \text{for all } x \in (a, b) \text{ with } x \neq x_0.$$

Hence, $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} f(x_0) + \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} \cdot \lim\limits_{x \to x_0} (x - x_0) = f(x_0)$. $\square$

Corollary: Suppose that $a < b$ and $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then f is continuous on (a, b).

Proof: The proof of this follows directly from the previous theorem. $\square$

Important: The converse of the previous theorem is false, ie: there exist everywhere continuous functions that are nowhere differentiable. Apparently the first such example was discovered by Bolzano around 1825, but was never published. The first published example was due to Weierstrass in 1875, (see below). The following function is everywhere continuous but nowhere differentiable.

$$W(x) := \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n} \qquad \text{for all } x \in \mathbb{R}.$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by,

$$f(x) := \begin{cases} x \sin(1/x) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

Show that the function f is continuous but **not** differentiable at $x_0 = 0$.

Answer: It follows from the sandwich theorem (for functions) that f is continuous at 0. We now show that f is not differentiable at $x_0 = 0$.

$$\lim_{x\to 0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \sin(1/x); \text{which does not exist.}$$

Hence f is not differentiable at $x_0 = 0$. □

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by,

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at $x_0 = 0$ and $f'(x_0) = 0$.

Suppose that $a < b$ and $f : (a, b) \to \mathbb{R}$. We say that a real number L is the *right-hand derivative* of f at a point $x_0 \in (a, b)$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon \qquad \text{for all } x \in (a, b) \text{ with } x_0 < x < x_0 + \delta.$$

We say that a real number L is the *left-hand derivative* of f at x_0 if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon \qquad \text{for all } x \in (a, b) \text{ with } x_0 - \delta < x < x_0.$$

We shall denote by $f'_+(x_0)$ the right-hand derivative of f at x_0 and by $f'_-(x_0)$ the left-hand derivative of f at x_0. Hence in this notation we have that,

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Theorem: *Suppose that $a < b$ and $f : (a, b) \to \mathbb{R}$. Then the function f is differentiable at a point $x_0 \in (a, b)$ if, and only if, both the left-hand and right-hand derivatives of f at x_0 exist and $f'_-(x_0) = f'_+(x_0)$.*

Proof: The proof of this result is left as an exercise for the reader. □

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by, $f(x) := |x|$. Show that f is **not** differentiable at $x_0 = 0$.

Answer: It is easy to see that both $f'_-(0)$ and $f'_+(0)$ exist, but that $f'(0)$ does not since, $-1 = f'_-(0) \neq f'_+(0) = 1$. $\square$

Rules for differentiation

The basic rules for differentiation should be familiar to you from first year. However, below we present the details of their proofs.

Theorem: *Suppose that $a < b$ and $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are differentiable at a point $x_0 \in (a, b)$ then;*

(i) $(f + g)'(x_0) = f'(x_0) + g'(x_0)$;

(ii) $(c \cdot f)'(x_0) = c \cdot f'(x_0)$;

(iii) $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$;

(iv) $(f \div g)'(x_0) = [f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)]/g^2(x_0)$, provided $g(x_0) \neq 0$.

Proof: Part (i) follows from,

$$\begin{aligned}
(f+g)'(x_0) &= \lim_{x\to x_0} \frac{[f(x)+g(x)] - [f(x_0)+g(x_0)]}{x - x_0} \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right) \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(x_0) + g'(x_0).
\end{aligned}$$

Part (ii) is very simple and is left as an exercise for the reader.

Part (iii) follows from,

$$\begin{aligned}
(f \cdot g)'(x_0) &= \lim_{x\to x_0} \frac{f(x)\cdot g(x) - f(x_0)\cdot g(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot g(x) + f(x_0) \cdot \frac{g(x) - g(x_0)}{x - x_0} \right) \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x\to x_0} g(x) + f(x_0) \cdot \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0).
\end{aligned}$$

We shall prove part (iv) by proving that the derivative of $1/g$ at $x = x_0$ is given by, $-g'(x_0)/g^2(x_0)$. The result will then follow from part (iii). Now,

$$\begin{aligned}
(1/g)'(x_0) &= \lim_{x\to x_0} \frac{1/g(x) - 1/g(x_0)}{x - x_0} = \lim_{x\to x_0} \left(\frac{g(x) - g(x_0)}{x - x_0} \cdot \frac{-1}{g(x)g(x_0)} \right) \\
&= \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \cdot \lim_{x\to x_0} \frac{-1}{g(x)g(x_0)} \\
&= -\frac{g'(x_0)}{g^2(x_0)}. \quad \square
\end{aligned}$$

Theorem: (Chain Rule) *Let* $f : A \to B$ *and* $g : B \to \mathbb{R}$ *be defined on non-empty open subsets* A *and* B *of* $\mathbb{R}$. *If* f *is differentiable at* $x_0 \in A$ *and* g *is differentiable at* $f(x_0)$ *then* $g \circ f$ *is differentiable at* x_0 *and* $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.

Proof: Define $H : B \to \mathbb{R}$ by,

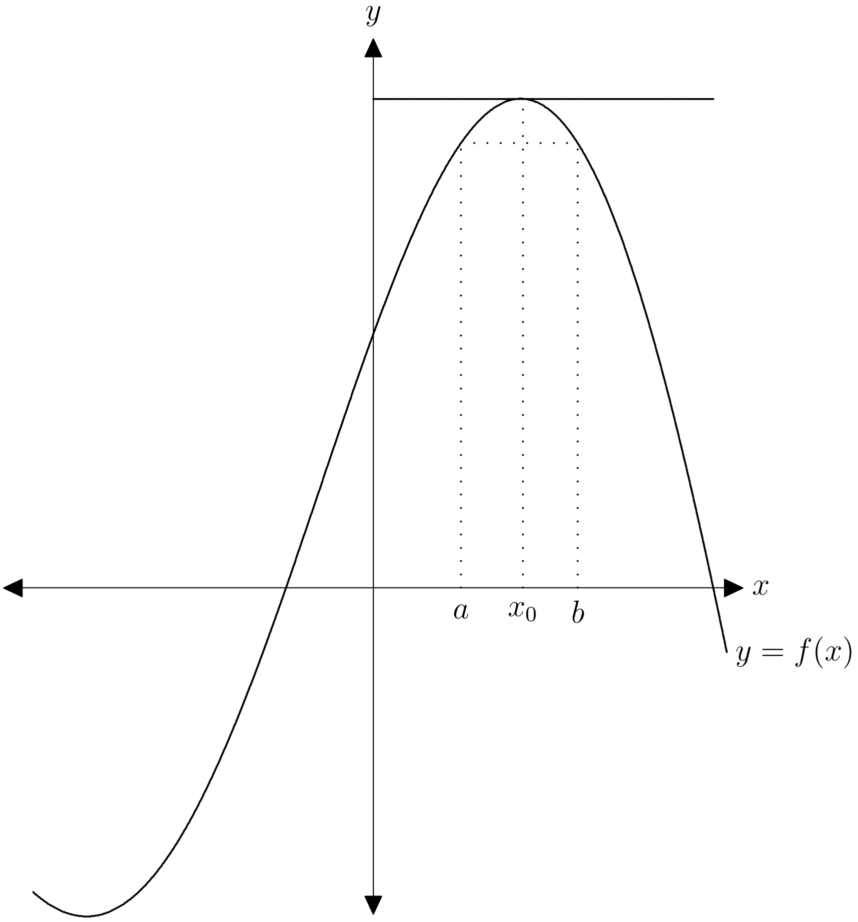
$$H(y) := \begin{cases} \dfrac{g(y) - g(f(x_0))}{y - f(x_0)} & y \neq f(x_0); \\ g'(f(x_0)) & y = f(x_0). \end{cases}$$

Now $\lim\limits_{y \to f(x_0)} H(y) = g'(f(x_0)) = H(f(x_0))$ and so H is continuous at $f(x_0)$. Moreover, we have that $(g \circ f)(x) - (g \circ f)(x_0) = H(f(x)) \cdot [f(x) - f(x_0)]$ for all $x \in A$ and so,

$$\begin{aligned} (g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{H(f(x)) \cdot [f(x) - f(x_0)]}{x - x_0} \\ &= \lim_{x \to x_0} H(f(x)) \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0)) \cdot f'(x_0). \quad \square \end{aligned}$$

Properties of differentiable functions

Theorem: (Rolle's Theorem) *Suppose that* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$. *If* $f(a) = f(b)$ *and* f *is differentiable on* (a, b) *then there exists a point* $x_0 \in (a, b)$ *such that* $f'(x_0) = 0$.



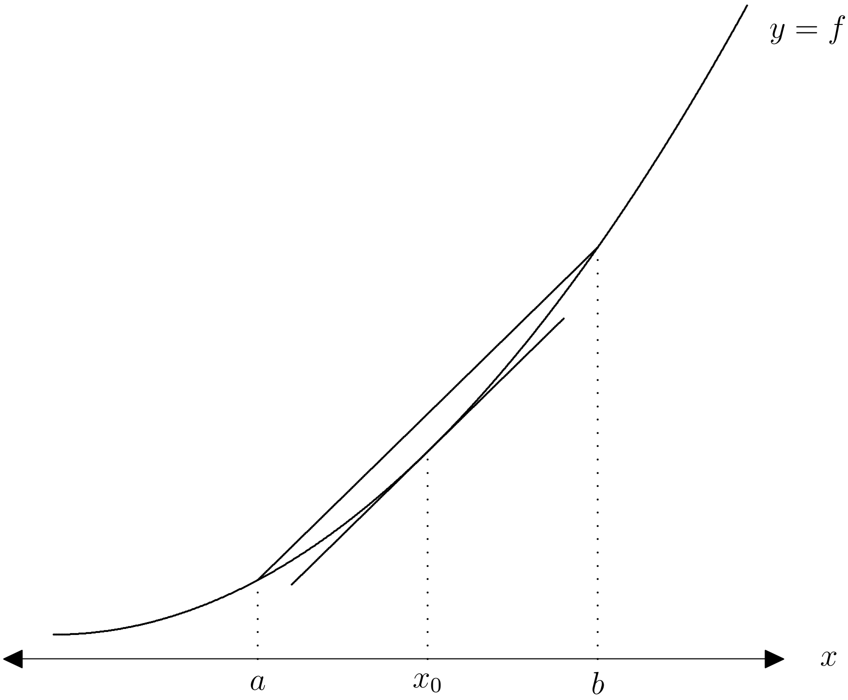
Proof: If f is constant on $[a,b]$ then $f'(x) \equiv 0$ on (a,b). Hence we may suppose that f is not constant on $[a,b]$. Therefore, either (i) $\max\{f(x) : x \in [a,b]\} > f(a) = f(b)$ or (ii) $\min\{f(x) : x \in [a,b]\} < f(a) = f(b)$. We shall only consider case (i) as case (ii) is similar. Choose $x_0 \in (a,b)$ such that $f(x_0) = \sup\{f(x) : x \in [a,b]\}$. Then since $f(x) \leqslant f(x_0)$ for all $x \in [a,b]$, $(f(x) - f(x_0))/(x - x_0) \leqslant 0$ for all $x \in [a,b]$ with $x_0 < x$. Therefore, $f'_+(x_0) = \lim_{x \to x_0^+}(f(x) - f(x_0))/(x - x_0) \leqslant 0$. On the other hand, $f(x) - f(x_0) \leqslant 0$ for all $x \in [a,b]$ with $x < x_0$ and so $(f(x) - f(x_0))/(x - x_0) \geqslant 0$ for all $x \in [a,b]$ with $x < x_0$. Therefore, $f'_-(x_0) = \lim_{x \to x_0^-}(f(x) - f(x_0))/(x - x_0) \geqslant 0$. Now since f is differentiable at x_0, $f'_+(x_0) = f'_-(x_0)$ and so $f'(x_0) = f'_+(x_0) = f'_-(x_0) = 0$. □

Exercise: Let $f : [-1,1] \to \mathbb{R}$ be defined by, $f(x) := (x+1)^m(x-1)^n$. Show that $f'(x_0) = 0$, where $x_0 := (m-n)/(m+n)$.

Example: Consider the function $f : [-1,1] \to \mathbb{R}$ defined by, $f(x) := |x|$. Show that $f'(x)$ never equals 0. Does this contradict Rolle's theorem?

Answer: First, note that $f'(0)$ does not exist and $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. Therefore, $f'(x)$ never equals 0. However, this does not contradict Rolle's theorem, because although f is continuous on $[-1,1]$ and $f(-1) = f(1)$, f is **not** differentiable on $(-1,1)$. □

Theorem: (Mean Value Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If f is differentiable on (a,b) then there a point $x_0 \in (a,b)$ such that* $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$.



Proof: Let $h : [a,b] \to \mathbb{R}$ be the equation of the line joining the points $(a, f(a))$ and $(b, f(b))$, ie: $h(x) := \dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)$.

Now let us consider the function $g : [a,b] \to \mathbb{R}$ defined by $g(x) := f(x) - h(x)$, ie: $g(x)$ is the difference between $f(x)$ and the line described by h. Then g is continuous on $[a,b]$, differentiable on (a,b) and $g(a) = g(b)$. Therefore by Rolle's theorem there exists a point $x_0 \in (a,b)$ such that $g'(x_0) = f'(x_0) - h'(x_0) = 0$, ie: $f'(x_0) = h'(x_0) = \dfrac{f(b) - f(a)}{b - a}$. $\square$

Example: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that f is increasing if, and only if, $f'(x) \geqslant 0$ for all $x \in \mathbb{R}$.

Answer: Suppose that $f'(x) \geqslant 0$ for all $x \in \mathbb{R}$. Let a, b be any real numbers such that $a < b$; we need to show that $f(a) \leqslant f(b)$. First we note that f is continuous on $[a,b]$ and differentiable on (a,b). Therefore, by the mean value theorem there exists a point $x_0 \in (a,b)$ such that $f(b) - f(a) = f'(x_0) \cdot (b - a) \geqslant 0$. From this it follows that $f(a) \leqslant f(b)$. Conversely, suppose that f is increasing on $\mathbb{R}$. Let x_0 be any point in $\mathbb{R}$ and let $x_0 < x$. Then $f(x) \geqslant f(x_0)$, ie: $f(x) - f(x_0) \geqslant 0$. Therefore, $\dfrac{f(x) - f(x_0)}{x - x_0} \geqslant 0$ for all $x > x_0$ and so $f'(x_0) = f'_+(x_0) = \lim\limits_{x \to x_0^+} \dfrac{f(x) - f(x_0)}{x - x_0} \geqslant 0$. $\square$

Exercise: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that f is strictly increasing if $f'(x) > 0$ for all $x \in \mathbb{R}$.

We now show that the derivative of an (everywhere) differentiable function satisfies an intermediate value property similar to that satisfied by continuous functions, despite the fact that f' may not be continuous.

Lemma: *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is differentiable at both a and b. If,*

(i) $f'(a) > 0$ then there exists a $\delta > 0$ such that $f(x) > f(a)$ for all $x \in [a,b]$ with $a < x < a + \delta$;

(ii) $f'(b) < 0$ then there exists a $\delta > 0$ such that $f(x) > f(b)$ for all $x \in [a,b]$ with $b - \delta < x < b$.

Proof: We shall only consider case (i) as case (ii) is similar. Since $f'(a) > 0$ there exists a $\delta > 0$ such that $(f(x) - f(a))/(x - a) > 0$ for all $a < x < a + \delta$. Therefore, for all $x \in [a,b]$ with $a < x < a + \delta$, $f(x) - f(a) = \dfrac{f(x) - f(a)}{x - a} \cdot (x - a) > 0$. $\square$

Theorem: (Darboux's Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$. If $f'(a) > k > f'(b)$ (or vice versa) then there exists a point $x_0 \in (a,b)$ such that $f'(x_0) = k$.*

Proof: Suppose that $f'(a) > k > f'(b)$ and consider the function $g : [a,b] \to \mathbb{R}$ defined by, $g(x) := f(x) - kx$. Since g is continuous on $[a,b]$, g attains its maximum value at some point $x_0 \in [a,b]$. Moreover, since $g'(a) > 0$ and $g'(b) < 0$ it follows from the previous Lemma that $x_0 \in (a,b)$. The result now follows as in Rolle's theorem by showing that $g'(x_0) = 0$. The proof for the case $f'(a) < k < f'(b)$ is similar. $\square$

Theorem: (Inverse Mapping Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$. If $f'(x) > 0$ for all $x \in (a,b)$ then f^{-1} is differentiable on $(f(a), f(b))$ and $(f^{-1})' = 1/(f' \circ f^{-1})$.*

Proof: Let $y_0 \in (f(a), f(b))$ and define $H : [a,b] \to \mathbb{R}$ by,

$$H(x) := \begin{cases} \dfrac{x - f^{-1}(y_0)}{f(x) - f(f^{-1}(y_0))} & x \neq f^{-1}(y_0); \\ \dfrac{1}{f'(f^{-1}(y_0))} & x = f^{-1}(y_0). \end{cases}$$

Then H is well defined since f is 1-to-1. In fact,

$$\lim_{x \to f^{-1}(y_0)} H(x) = 1/f'(f^{-1}(y_0)) = H(f^{-1}(y_0)) \text{ and so } H \text{ is continuous at } f^{-1}(y_0).$$

Moreover from our eariler work on continuity we know that f^{-1} is continuous on $[f(a), f(b)]$. Therefore,

$$\begin{aligned} 1/f'(f^{-1}(y_0)) = H(f^{-1}(y_0)) &= H\left(\lim_{y \to y_0} f^{-1}(y)\right) \\ &= \lim_{y \to y_0} H(f^{-1}(y)) \\ &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = (f^{-1})'(y_0) \quad \square \end{aligned}$$

Remark: The hypothesis that $f'(x) > 0$ for all $x \in (a,b)$ is essential. In fact if f is strictly increasing and differentiable on (a,b) but $f'(x_0) = 0$ for some $x_0 \in (a,b)$ then the inverse, f^{-1} is **not** differentiable at $f(x_0)$. Indeed, if f^{-1} were differentiable at $f(x_0)$ then by the chain rule we would have, $1 = (f^{-1} \circ f)'(x_0) = (f^{-1})'(f(x_0)) \cdot f'(x_0) = 0$; which is impossible. Therefore, f^{-1} cannot be differentiable at $f(x_0)$. The function $f(x) := x^3$ is strictly increasing and differentiable on $\mathbb{R}$ however $x \mapsto \sqrt[3]{x}$ (the inverse of f) is not differentiable at $f(0) = 0$.

Theorem: (Cauchy's Mean Value Theorem). *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \neq 0$ for all $x \in (a,b)$ then there exists a point $x_0 \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.$$

Proof: Consider the auxiliary function $h : [a,b] \to \mathbb{R}$ defined by,

$$h(x) := \begin{vmatrix} f(x) & g(x) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix}$$

Now h is continuous on $[a,b]$ and differentiable on (a,b) and $h(a) = h(b) = 0$ so by Rolle's theorem there exists a point $x_0 \in (a,b)$ such that $h'(x_0) = 0$; that is,

$$\begin{vmatrix} f'(x_0) & g'(x_0) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0$$

ie: $f'(x_0)\cdot[g(b)-g(a)] = g'(x_0)\cdot[f(b)-f(a)]$. Now since $g'(x)\neq 0$ for all $x\in(a,b)$, Rolle's theorem tells us that $g(b)-g(a)\neq 0$ and so the result follows. $\square$

Remark: Cauchy's mean value theorem has a geometric interpretation. If we consider the curve defined by the parametric equation $\boldsymbol{x} := (f(t),g(t))$, $t\in[a,b]$. Then the conclusion of the theorem is that there exists a point $(f(x_0),g(x_0))$ on the curve such that the slope, $g'(x_0)/f'(x_0)$ of the tangent line to the curve at that point is equal to the slope of the line segment joining the end points of the curve.

Theorem: (Taylor's Theorem) *Suppose that* $a<b$ *and* $f:[a,b]\to\mathbb{R}$. *If* $f^{(n)}$ *is continuous on* $[a,b]$ *and differentiable on* (a,b) *then for each* $x\in(a,b]$ *there exists a point* $\zeta\in(a,x)$ *such that* $f(x)=P_n(x)+R_n(x)$, *where*

$$P_n(x) := f(a)+\sum_{k=1}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} \quad \textit{and} \quad R_n(x) := f^{(n+1)}(\zeta)\frac{(x-a)^{n+1}}{(n+1)!}.$$

Proof: Fix $x_0\in(a,b]$ and define $M\in\mathbb{R}$ by,

$$f(x_0)=f(a)+\sum_{k=1}^{n} f^{(k)}(a)\frac{(x_0-a)^k}{k!}+M\cdot\frac{(x_0-a)^{n+1}}{(n+1)!}$$

We need to show that there exists a $\zeta\in(a,x_0)$ such that $f^{(n+1)}(\zeta)=M$. Consider the auxiliary function $g:[a,x_0]\to\mathbb{R}$ defined by,

$$g(x) := -f(x_0)+f(x)+\sum_{k=1}^{n} f^{(k)}(x)\frac{(x_0-x)^k}{k!}+M\cdot\frac{(x_0-x)^{n+1}}{(n+1)!}.$$

Now g is continuous on $[a,x_0]$ and differentiable on (a,x_0) and $g(a)=g(x_0)=0$. Therefore, by Rolle's theorem there exists a point $\zeta\in(a,x_0)$ such that $g'(\zeta)=0$.

$$\begin{aligned} g'(x) &= f'(x)+\sum_{k=1}^{n}\left\{f^{(k+1)}(x)\frac{(x_0-x)^k}{k!}-f^{(k)}(x)\frac{(x_0-x)^{k-1}}{(k-1)!}\right\}-M\cdot\frac{(x_0-x)^n}{n!} \\ &= \frac{(x_0-x)^n}{n!}\cdot\{f^{(n+1)}(x)-M\} \quad \text{for all } x\in(a,x_0). \end{aligned}$$

Therefore, $f^{(n+1)}(\zeta)=M$. This completes the proof. $\square$

In Taylor's theorem the polynoimal $P_n(x)$ is called the n-th degree *Taylor polynomial* for f at a and $R_n(x)$ is called the *Lagrange remainder*.

Suppose that $a<b$ and $f:[a,b]\to\mathbb{R}$. If for each fixed $x\in[a,b]$,

$$\lim_{n\to\infty} R_n(x)=0.$$

Then for each $x\in[a,b]$, the power series, $\displaystyle\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$ converges to $f(x)$.

L'Hospital's Theorems

Recall that if $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \text{ provided } L_2 \neq 0.$$

We now look into the case when $L_1 = L_2 = 0$ (there's no point in considering the case $L_1 \neq 0$ and $L_2 = 0$ since the limit will be $\pm\infty$).

Theorem: *Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$. If $f(x_0) = g(x_0) = 0$, $g'(x_0) \neq 0$ and both $f'(x_0)$ and $g'(x_0)$ exist at some point $x_0 \in (a, b)$, then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

Proof: The trick in this proof is to multiply and divide by $(x - x_0)$.

$$\begin{aligned}
\lim_{x\to x_0} \frac{f(x)}{g(x)} &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \cdot \left(\frac{x - x_0}{g(x) - g(x_0)}\right) \\
&= f'(x_0)/g'(x_0). \quad \square
\end{aligned}$$

Theorem: *Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$. Let x_0 be any point in (a, b) such that $f(x_0) = g(x_0) = 0$ and $g'(x) \neq 0$ for all $x \neq x_0$. Then,*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}, \textit{ whenever the limit on the right exists}$$

Proof: By Cauchy's mean value theorem there exists for each $x \in (a, b)$ a point ζ_x between x_0 and x such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\zeta_x)}{g'(\zeta_x)}.$$

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)},$$

as ζ_x tends to x_0 as x tends to x_0. $\square$

Remark: The previous theorem is also true when $\lim_{x\to x_0} f(x) = \pm\infty$ and $\lim_{x\to x_0} g(x) = \pm\infty$ and may also be extended to the case when x_0 is replaced by $\pm\infty$.

Exercises

1. Calculate the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ defined by, $f(x) := x^3$.

2. Consider the function $f : (0, \infty) \to \mathbb{R}$ defined by, $f(x) := \log_e(x)$. Show that the inverse of f exists and is differentiable. Moreover, show that $(f^{-1})'(x) = f^{-1}(x)$ for all $x \in \mathbb{R}$. Note: the function f^{-1} is usually called the *exponential* function.

3. Let $f : (0, \infty) \to \mathbb{R}$ be defined by, $f(x) := x \cdot \log_e(1 + 1/x)$.

(a) Calculate $\lim_{x\to\infty} f(x)$. *Hint:* Consider the derivative of the function $g : (0, \infty) \to \mathbb{R}$ defined by, $g(x) := \log_e(x)$ at $x = 1$.

(b) Show that $\lim\limits_{n\to\infty} n \cdot \log_e(1 + 1/n) = 1$.

(c) By using the fact that the exponential function, $x \mapsto e^x$, is continuous show that $\lim\limits_{n\to\infty} (1 + 1/n)^n = e$.

4. Show that the function $f : [0, \pi/2] \to \mathbb{R}$ defined by, $f(x) := \sin(x)$ has a differentiable inverse. Moreover show that $(\sin^{-1})'(x) = 1/\sqrt{1 - x^2}$.

5. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that f is a constant function if, and only if, $f'(x) \equiv 0$ on (a, b).

6. Let $f : \mathbb{R} \to \mathbb{R}$ differentiable. Show that if $f' : \mathbb{R} \to \mathbb{R}$ is increasing on $\mathbb{R}$ then f' is continuous on $\mathbb{R}$. *Hint:* use the fact that an increasing function is continuous if, and only if, it satisfies the intermediate value property, (see the section on continuity).

7. Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. If f'' exists and is continuous on $\mathbb{R}$ and (i) $f'(x_0) = 0$; (ii) $f''(x_0) > 0$. Show that f has a local minimum at x_0. *Hint:* Consider the 1st order Taylor's expansion of f around x_0, (with remainder).

8. (Taylor's Theorem) Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$. If $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b) then for each $x_0 \in (a, b]$ there exists a point $\zeta \in (a, x_0)$ such that $f(x_0) = P_n(x_0) + R_n(x_0)$, where

$$P_n(x_0) := f(a) + \sum_{k=1}^{n} f^{(k)}(a)\frac{(x_0 - a)^k}{k!} \quad \text{and} \quad R_n(x_0) := f^{(n+1)}(\zeta)\frac{(x_0 - \zeta)^n(x_0 - a)}{n!}$$

Hint: Consider the auxiliary function $g : [a, b] \to \mathbb{R}$ defined by,

$$g(x) := -f(x_0) + f(x) + \sum_{k=1}^{n} f^{(k)}(x)\frac{(x_0 - x)^k}{k!} + M \cdot (x_0 - x).$$

9. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be differentiable on I. Suppose that $f(a) < 0 < f(b)$ and that there exist m, M such that $0 < m \leqslant f'(x) \leqslant M$ for all $x \in I$. Let $x_1 \in I$ be arbitrary and define $x_{n+1} := x_n - f(x_n)/M$ for all $n \in \mathbb{N}$. Show that the sequence $(x_n : n \in \mathbb{N})$ is well defined and converges to the unique zero $r \in I$ of f. *Hint:* if $\phi(x) := x - f(x)/M$, show that $0 \leqslant \phi'(x) \leqslant 1 - m/M < 1$ and that $\phi([a, b]) \subseteq [a, b]$.

10. Calculate the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{\tan^{-1}(x)}{x}$; (b) $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$; (c) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$; (d) $\lim\limits_{x\to 0} \dfrac{1 - \cos(x)}{x^2}$.

Appendix A

A *proposition* $\mathbf{P}$ is a statement which is either *true* or *false*. If the proposition is dependent upon a *variable* x which takes values in some non-empty set A then we denote it by $\mathbf{P}(x)$. To prove a statement of the form: "for every $x \in A$, $\mathbf{P}(x)$" is true. We must show that if x is any element of A then $\mathbf{P}(x)$ is true. Hence a proof for something like this should start: "Let x be any element of A" and then go on to show that $\mathbf{P}(x)$ is true for this value of $x \in A$.

To prove a statement like: "there exists an $x \in A$ such that $\mathbf{P}(x)$" is true we must produce/construct/guess an element x_0 of the set A such that $\mathbf{P}(x_0)$ is true. Hence a proof for something like this should start: "Set x equal to x_0" (for some particular value of $x_0 \in A$) and then go on to check/verify that $\mathbf{P}(x_0)$ is true. In practice we would write something like: "Set $x := x_0$" and then check/verify $\mathbf{P}(x_0)$ is true.

Given a sequence $(a_n : n \in \mathbb{N})$ and a real number L we may consider the proposition $\mathbf{P}(\varepsilon, N) :=$ "$|a_n - L| < \varepsilon$ for all $n > N$." We may also consider the proposition $\mathbf{Q}(\varepsilon) :=$ "there exists an $N \in \mathbb{N}$, such that $\mathbf{P}(\varepsilon, N)$." and the proposition $\mathbf{R} :=$ "for every $\varepsilon \in P$, $\mathbf{Q}(\varepsilon)$." That is, $\mathbf{R} :=$ "for every $\varepsilon \in P$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n > N$." Here, $P := \{x \in \mathbb{R} : 0 < x\}$.

So to prove that the real number L is the limit of the sequence $(a_n : n \in \mathbb{N})$ we need to show that $\mathbf{R}$ is true. Therefore from our previous discussion our proof of this fact should start something like: "Let ε be any positive real number. Set $N := N_0$" and then go on to check/verify that "$|a_n - L| < \varepsilon$ for all $n > N_0$" is true.

Since our choice of $N \in \mathbb{N}$ is (possibly) dependent upon ε we may sometimes write our choice of N_0 as $N(\varepsilon)$. So the problem of showing that L is the limit of $(a_n : n \in \mathbb{N})$ "boils down to" the following 3 steps.

Step 1. Finding a function, $\varepsilon \mapsto N(\varepsilon)$, such that $\mathbf{P}(\varepsilon, N(\varepsilon))$ is true for all $\varepsilon > 0$ (this step often involves algebraic manipulations and screwing around with inequalities).

Step 2. Writing: "Let ε be any positive real number. Set $N := N(\varepsilon)$ then,"

Step 3. Presenting a justification/verification that indeed, $\mathbf{P}(\varepsilon, N(\varepsilon))$ is true.

Step 1. is of course usually hardest step and is difficult to give a recipe for. However the first part of this course is devoted to giving some examples of how to carry out Steps 1, 2 and 3.