260 Test 2007F

① (a) $\frac{dy}{dt} = y^2 \cos t$

$$\int \frac{dy}{y^2} = \int \cos t \, dt$$

$$-\frac{1}{y} = \sin t + c$$

$$y = \frac{-1}{\sin t + c}$$

($y=0$ is a missing soln, but it does not satisfy initial condition).

$$1 = y(\pi/2) = \frac{-1}{1+c}$$

$\therefore \quad 1+c = -1 \quad$ ie $c = -2$

$$y(t) = \frac{1}{2 - \sin t}$$

(b) $\frac{dy}{dt} - 5y = e^t$

Linear. $\mu(t) = \exp\left(\int -5\,dt\right) = e^{-5t}$

$$e^{-5t}\frac{dy}{dt} - 5e^{-5t}y = e^{-4t}$$

$$\frac{d}{dt}\left(e^{-5t}y\right) = e^{-4t}$$

$$e^{-5t}y = -\frac{1}{4}e^{-4t} + c$$

$$y = -\frac{1}{4}e^{t} + ce^{5t}$$

(2) $$\frac{dy}{dt} = y^2 - t \qquad y(2) = 1$$

$$f(t,y) = y^2 - t \qquad t_0 = 2, \; y_0 = 1$$

(a) There is a unique solution.
f is continuous for all t, y + so at $(2,1)$
$\frac{\partial f}{\partial y} = 2y$ which is also continuous + so at $(2,1)$.

(b) $h = \frac{2.4 - 2}{2} = 0.2$

$t_0 = 2, \quad y_0 = 1, \quad f(t,y) = y^2 - t$

Step 1 $\quad m_1 = f(t_0, y_0) = -1$

$m_2 = f(t_0 + h, y_0 + hm_1) = f(2.2, 0.8)$
$= -1.56$

$$y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$$
$$= 1 + \frac{0.2}{2}(-1 - 1.56)$$
$$= 0.744$$

Step 2 $\quad m_1 = f(t_1, y_1) = -1.6465$

$m_2 = f(t_1 + h, y_1 + hm_1)$
$= f(2.4, .4147)$
$= -2.228$

$$y_2 = y_1 + \frac{h}{2}(m_1 + m_2) = 0.3565$$

(c) With 10 steps instead of 2, h would become $\frac{h}{5}$
Since Improved Euler has order 2
$$E\left(\frac{h}{5}\right) \approx C\left(\frac{h}{5}\right)^2 = \frac{Ch^2}{25} = \frac{E(h)}{25}$$
Expect the error to be 25 times less.

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③ (a) $\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{80}\right) - 0.5$

(b) First find the equilibrium solutions.

$$\frac{dP}{dt} = 0 \Leftrightarrow 0.05P\left(1 - \frac{P}{80}\right) - 0.5 = 0$$

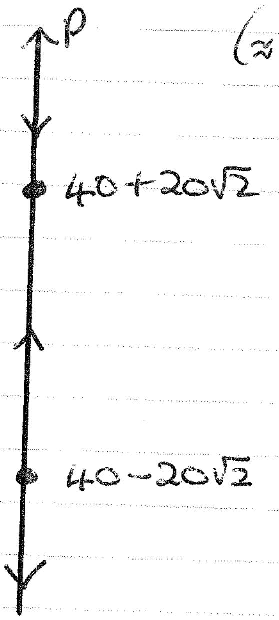
$$\Leftrightarrow 0.05P(80 - P) - 0.5 \times 80 = 0$$

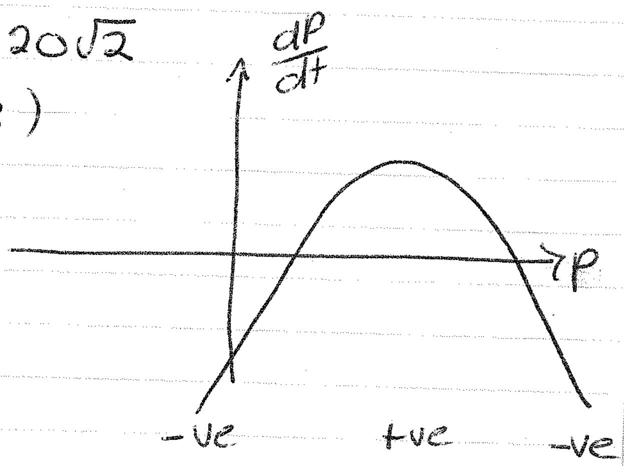
$$\Leftrightarrow -0.05P^2 + 4P - 40 = 0$$

$$\Leftrightarrow P = \frac{-4 \pm \sqrt{16 - 8}}{-0.1}$$

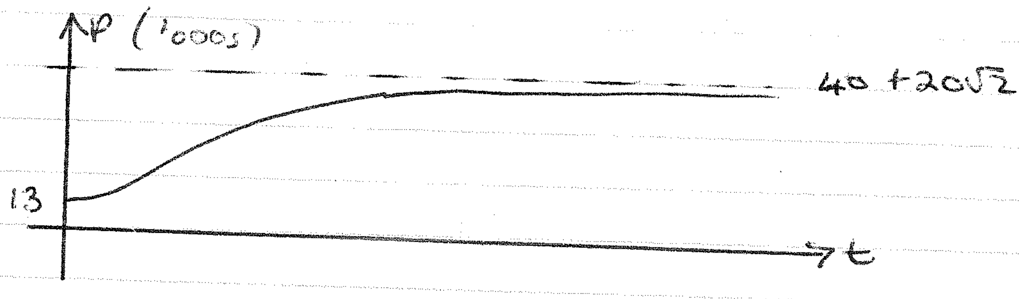
$$= 40 \pm 20\sqrt{2}$$

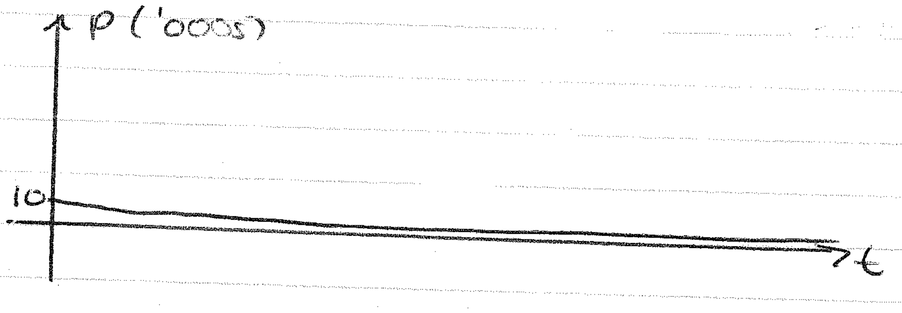
$(\approx 11.7 \text{ \& } 68.3)$





(c)
(i)

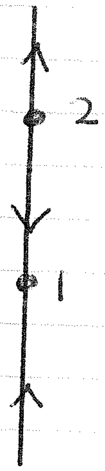




4 $\dfrac{dy}{dt} = (y-1)(y-\alpha)$

(a) $\dfrac{dy}{dt} = (y-1)(y-2)$

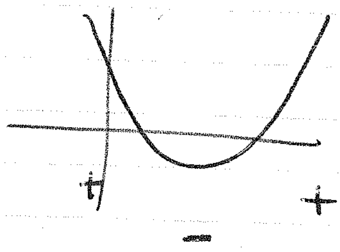
$= 0$ when $y=1,\ y=2$



$f(y) = y^2 - 3y + 1$

$\dfrac{\partial f}{\partial y} = 2y - 3$

OR



$\left.\dfrac{\partial f}{\partial y}\right|_1 = -1 \quad \therefore$ sink

$\left.\dfrac{\partial f}{\partial y}\right|_2 = 1 \quad \therefore$ source

OR Draw phase line from above quadratic + see that 1 is sink + 2 is source.

(b) Equilibrium when $y=1$ or $y=\alpha$.

$f(y) = y^2 - (\alpha+1)y + \alpha, \quad \dfrac{\partial f}{\partial y} = 2y - (\alpha+1)$

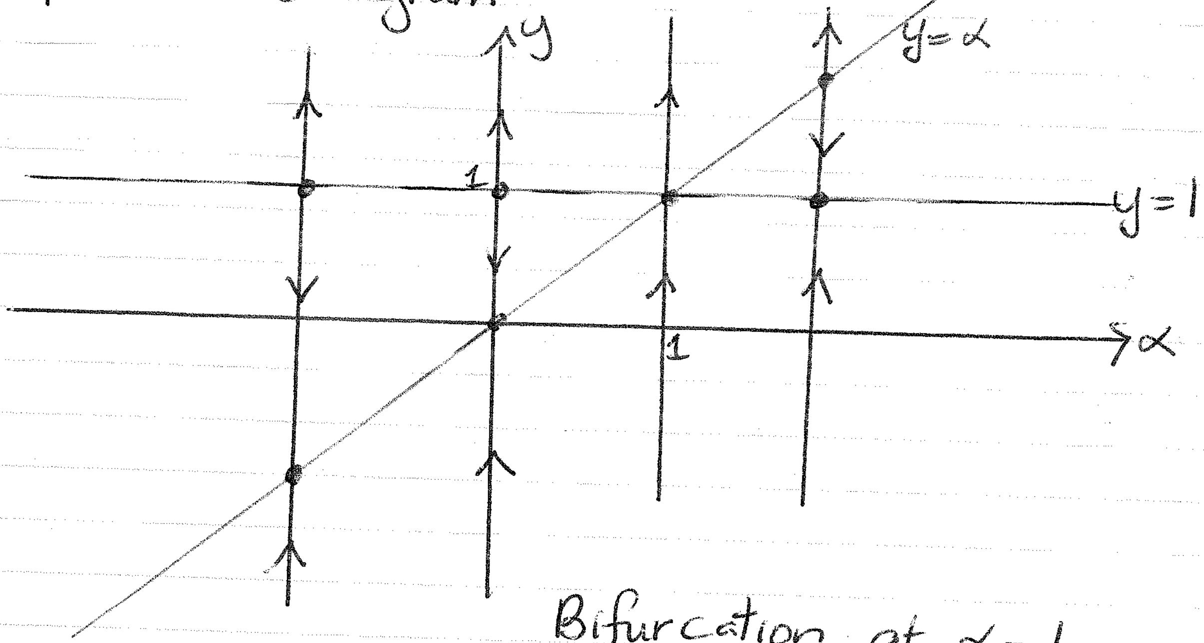
2 equilibria if $\alpha \neq 1$, otherwise 1.

$\left.\dfrac{\partial f}{\partial y}\right|_1 = 1-\alpha$ source $\alpha < 1$, sink $\alpha > 1$

$\left.\dfrac{\partial f}{\partial y}\right|_\alpha = \alpha - 1$ sink $\alpha < 1$, source $\alpha > 1$.

If $\alpha = 1$, ∂ $f(y) = (y-1)^2$ which is always positive when $y \neq 1$ $\therefore$ node.

Bifurcation diagram



Bifurcation at $\alpha = 1$.

(5) At equilibrium

$$\left.\begin{array}{l} x+z=0 \\ x^3+y=0 \\ y-z=0 \end{array}\right\}$$

ie $$x=-z$$
$$y=z$$

$$0 = x^3 + y = -z^3 + z = z(1-z^2)$$
$$= z(1-z)(1+z)$$

$z = 0$ or 1 or -1

When $z=0$, $x=y=0$

When $z=1$, $x=-1$, $y=1$

When $z=-1$, $x=1$, $y=-1$

Equilibrium solutions are $(0,0,0)$
$(-1,1,1)$
$(1,-1,-1)$

(6) $\qquad$ 260 07F

$$\frac{dx}{dt} = y^2 = f(t,x,y)$$
$$\frac{dy}{dt} = t - x = g(t,x,y)$$

(a) f & g are both continuous at $(0,1,2)$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial g}{\partial x} = -1, \quad \frac{\partial g}{\partial y} = 0$$

These are all continuous at $(0,1,2)$

$\therefore$ there is a unique solution.

(b) $h = \frac{0.1 - 0}{1} = 0.1$

$$x(0.1) \approx x(0) + h f(t_0, x_0, y_0)$$
$$= 1 + 0.1 \times 4$$
$$= 1.4$$

$$y(0.1) \approx y(0) + h g(t_0, x_0, y_0)$$
$$= 2 + 0.1(0 - 1)$$
$$= 1.9$$