Department of Mathematics

MATHS 260 Differential Equations
Mid-semester Test 2004S - Solutions

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1. (7 marks) Find a solution to the following initial value problem:

$$\frac{dy}{dt} = \frac{1+2y}{t^2}, \quad y(1) = 0.$$

There are 2 ways to solve this equation:

(1) Separable eqn $\quad \displaystyle\int \frac{dy}{1+2y} = \int \frac{dt}{t^2}$ for $y \neq -\frac{1}{2}$

i.e. $\frac{1}{2}\ln|1+2y| = -\frac{1}{t} + c, \quad$ c arbitrary

$\Rightarrow \quad |1+2y| = \tilde{k}\exp\left(-\frac{2}{t}\right) \quad \tilde{k} = e^{2c} > 0$

$\Rightarrow \quad 1+2y = k\exp\left(-\frac{2}{t}\right) \quad k \neq 0$

$\Rightarrow \quad y = -\frac{1}{2} + k\exp\left(-\frac{2}{t}\right) \quad k \neq 0$

(in fact $y(t) = -\frac{1}{2}$ is a soln, ie. picking $k=0$ yields a soln, but we cannot deduce this from the working above)

Use the initial condition: $y(1)=0 \Rightarrow 0 = -\frac{1}{2} + ke^{-2}$

$\Rightarrow k = \frac{1}{2}e^2$

so $y(t) = -\frac{1}{2} + \frac{1}{2}e^{2(1-\frac{1}{t})}$ solves the IVP.

Alternatively

(2) Linear eqn $\quad \displaystyle\frac{dy}{dt} - \frac{2}{t^2}y = \frac{1}{t^2}$

$a(t) = -\frac{2}{t^2}, \quad \mu(t) = \exp\left(\int a(t)\,dt\right)$

$= \exp\left(\int -\frac{2}{t^2}\,dt\right)$

$= \exp\left(\frac{2}{t}\right)$

Multiply eqn by $\mu(t)$:

$$e^{2/t}\frac{dy}{dt} - \frac{2}{t^2}e^{2/t}y = \frac{1}{t^2}e^{2/t}$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{2/t}y\right) = \frac{1}{t^2}e^{2/t}$$

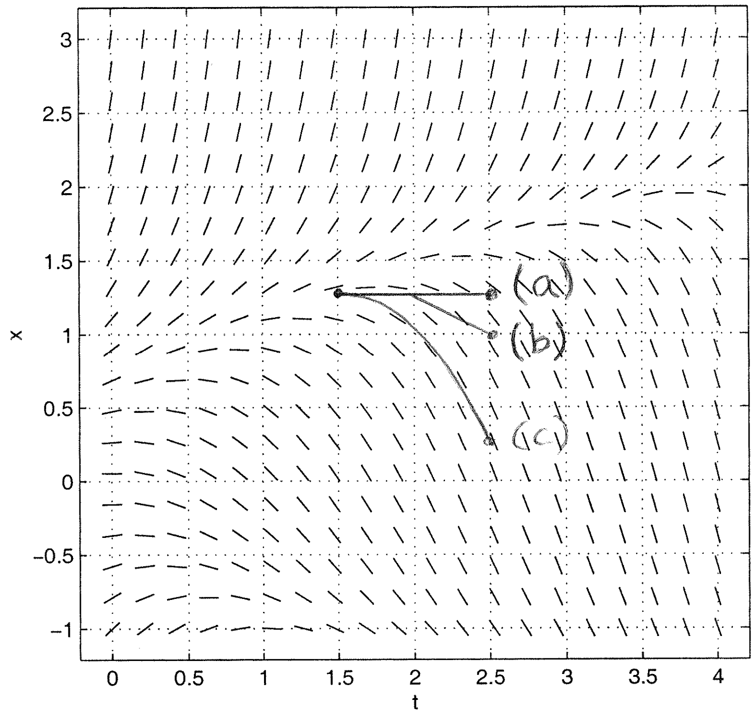
$$\Rightarrow \quad e^{2/t}y = \int \frac{1}{t^2}e^{2/t}\,dt$$

$$= -\frac{1}{2}e^{2/t} + c \qquad c \text{ arbitrary constant}$$

$$\Rightarrow \quad y(t) = -\frac{1}{2} + ce^{-2/t} \qquad c \text{ arbitrary}$$

Use initial condition as in soln to separable eqn above.

2. (8 marks)
The following picture shows the slope field for a differential equation.



(a) On this picture, carefully draw the solution you would obtain if you used one step of Euler's method with $h = 1$ to approximate at $t = 2.5$ the solution to the differential equation satisfying the initial condition $x(1.5) = 1.25$.

(b) On the same picture, carefully draw the solution you would obtain if you used two steps of Euler's method with $h = 0.5$ to approximate the same solution.

(c) Estimate the errors in the approximate solutions you obtained in (a) and (b) at $t = 2.5$.

(d) Hence estimate the stepsize you would need to use if you wanted to use Euler's method to approximate the solution at $t = 2.5$ with an error of less than 1%.

(a) Straight line marked (a) in diagram above is the Euler solution with $h=1$. See $y(2.5) \approx 1.25$

(b) Two line segments ending at point (b) in diagram above show the Euler solution with $h = 0.5$.
Get $y(2.5) \approx 1.0$

(c) The curve marked (c) in the diagram shows the approximate solution sketched using the direction field. Find $y(2.5) \approx 0.3$

Hence error in soln in (a) is approx $|1.25 - 0.3|$

$= 0.95$

error in soln in (b) is approx $|1.0 - 0.3|$

$= 0.7$

(d) At $t = 2.5$ $y(t) \approx 0.3$ (from (c)).

1% error at $t = 2.5$ means an error of $\lesssim 0.003$

Error with $h = 0.5$ is ≈ 0.7

Need to reduce the error by a factor of $\frac{0.7}{0.003} \approx 233$

Since Euler's method is order 1, we need to reduce the stepsize by the same factor. Thus a stepsize of $\approx \frac{0.5}{233} \approx 0.002$ would be appropriate.

Note: another way to answer (c) and (d) is to note that for an order 1 method, error with stepsize h is approx twice the error with stepsize $\frac{h}{2}$ so we can approximate $\left|\text{error}\left(\frac{h}{2}\right)\right|$ by $\left|\text{error}(h) - \text{error}\left(\frac{h}{2}\right)\right| = 0.25$.

To reduce this error to 1% of 0.3, ie. 0.003 we need to reduce stepsize by a factor of $\approx \frac{0.25}{0.003} \approx 85$.

Hence a stepsize of $\frac{0.5}{85} \approx 0.005$ would be appropriate.

3. (14 marks)

Consider the following differential equation:

$$\frac{dy}{dt} = y(y-2) + \mu$$

(a) Find all equilibrium solutions and determine their types (e.g., sink, node).

(b) Draw the bifurcation diagram. Identify any values of μ for which a bifurcation exists.

(c) Describe the behaviour of solutions as t increases in the case $\mu = 0$.

(a) Equilibria satisfy $y(y-2) + \mu = 0$

i.e. $y^2 - 2y + \mu = 0$

or $y = \dfrac{2 \pm \sqrt{4 - 4\mu}}{2} = 1 \pm \sqrt{1-\mu}$ $(\mu \le 1)$

If $\mu > 1$ there are no real solns

If $\mu = 1$ there is one real soln

If $\mu < 1$ there are two real solns.

$$\frac{\partial f}{\partial y} = 2y - 2$$

so $\left.\dfrac{\partial f}{\partial y}\right|_{1 \pm \sqrt{1-\mu}} = 2(1 \pm \sqrt{1-\mu}) - 2 = \pm\sqrt{1-\mu}$

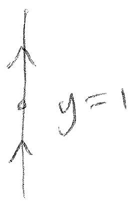
Hence $y = 1 + \sqrt{1-\mu}$ is a source for $\mu < 1$

$y = 1 - \sqrt{1-\mu}$ is a sink for $\mu < 1$

At $\mu = 1$, linearisation is unhelpful since $\dfrac{\partial f}{\partial y} = 0$ at the equilibrium.

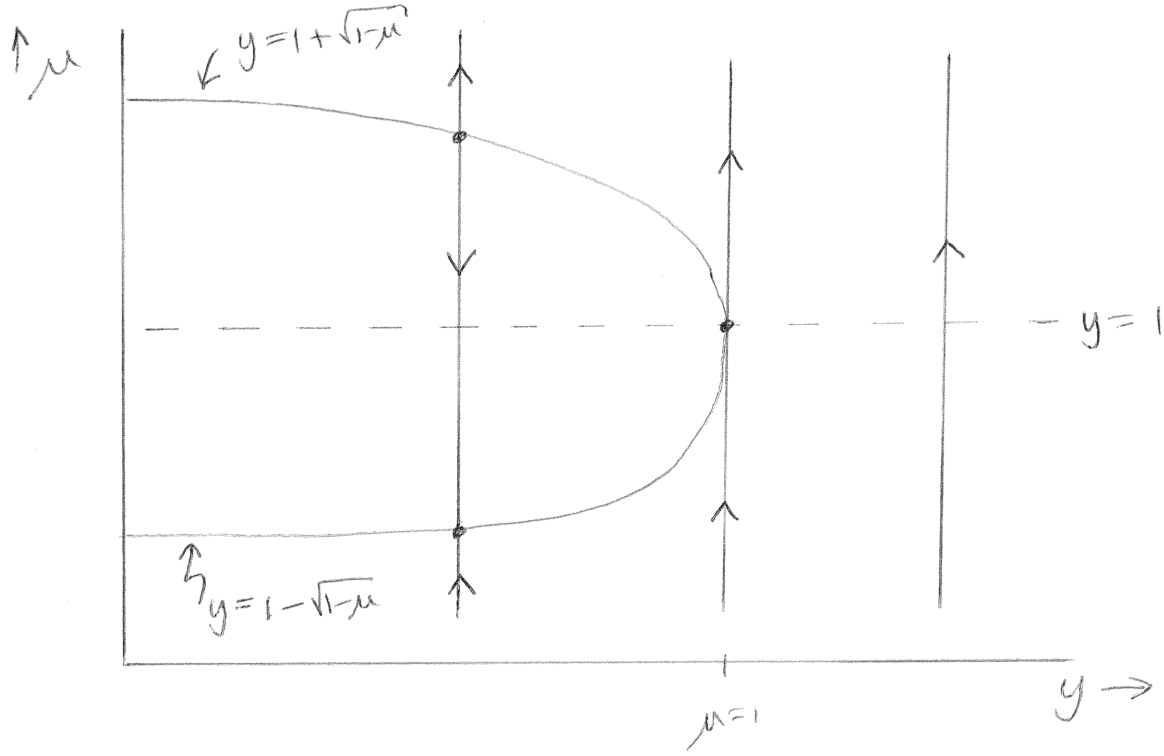
However, at $\mu = 1$, $\dfrac{dy}{dt} = y^2 - 2y + 1 = (y+1)^2 \geq 0$

Hence the phase line at $\mu = 1$ is:

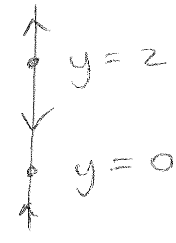


so $y = 1$ is a node a $\mu = 1$

(b) There is a bifurcation at $\mu = 1$



(c) At $\mu = 0$, the equilibria are at $y = 0$, $y = 2$ and the phase line is



(from bifurcation diagram)

Hence, if $y(0) = 0$ then $y(t) = 0$ for all t

if $y(0) = 2$ then $y(t) = 2$ for all t

if $y(0) < 0$ or if $0 < y(0) < 2$ then $y(t) \to 0$ as $t \to \infty$

if $y(0) > 2$ then $y(t) \to \infty$ as t increases.

4. (16 marks)

Consider the following system of differential equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Find a solution to this system of equations. Your answer should contain two arbitrary constants.

(b) What are the straight line solutions to this system?

(c) Find the solution that passes through $(x, y) = (1, 0)$ at $t = 0$. Express your solution in the form $(x(t), y(t))$. Show that this solution satisfies $y = -x + x^2$ for positive x.

(d) What can you say about the long term behaviour of all solutions?

(e) The picture below shows the slope field for the system of equations. On this picture:

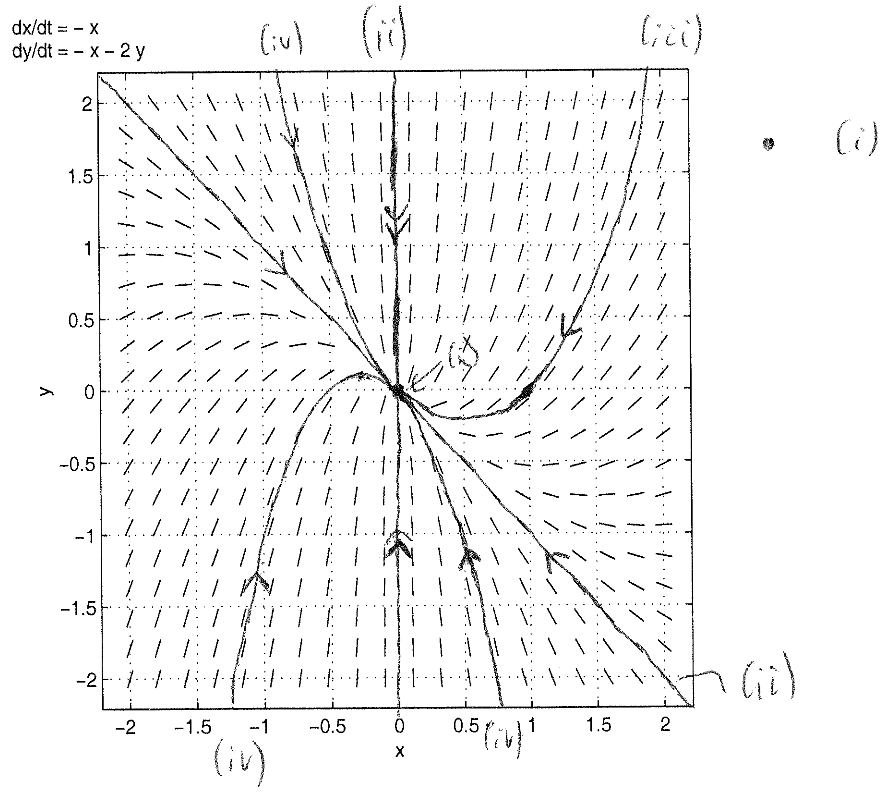
i. identify all equilibrium solutions;

ii. draw the straight line solutions you found in part (b) above;

iii. draw the solution you found in part (c) above;

iv. sketch three other solution curves.

For each solution curve you draw, you should draw an arrow indicating the direction moved as time increases.



(blank page for your working)

4. (a) There are two ways to solve this system

1. $\frac{dx}{dt} = -x$ gives $\int \frac{dx}{x} = \int -dt$

$$\ln|x| = -t + k$$

$$x = c_1 e^{-t}$$

so $\frac{dy}{dt} = -x - 2y = -c_1 e^{-t} - 2y$

$$\frac{dy}{dt} + 2y = -c_1 e^{-t}$$

use integrating factor with $a(t) = 2$ so $\mu(t) = e^{\int a(t)\,dt} = e^{2t}$

$$e^{2t}\frac{dy}{dt} + 2ye^{2t} = -c_1 e^{t}$$

$$\frac{d}{dt}\left[y e^{2t} \right] = -c_1 e^{t}$$

$$y e^{2t} = -c_1 e^{t} + c_2$$

$$y = -c_1 e^{-t} + c_2 e^{-2t}.$$

2. eigenvalues and eigenvectors:

$$\det\begin{pmatrix} -1-\lambda & 0 \\ -1 & -2-\lambda \end{pmatrix} = (-1-\lambda)(-2-\lambda) + 0 = \lambda^2 + 3\lambda + 2 =$$

$(\lambda+1)(\lambda+2) = 0$ so $\lambda = -1$ and $\lambda = -2$ are e-values.

e-vectors: $\lambda = -1$: $\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $-x-y=0$ so $y = -x$

so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an e-vector

$\lambda = -2$: $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $x = 0$ so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} =$

so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an e-vector

this means that the solution is $c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (i

$e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$ and $e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$ are straight line solutions.

(c) substitute $t=0$ in $(*)$ must equal $(1,0)$, so:

$$\begin{pmatrix}c_1\\-c_1\end{pmatrix}+\begin{pmatrix}0\\c_2\end{pmatrix}=\begin{pmatrix}c_1\\-c_1+c_2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

so $c_1=1$

$-c_1+c_2=0$

$c_2=1.$

So

$x(t)=+e^{-t}$

$y(t)=-e^{-t}+e^{-2t}$

and

$$-x+x^2=-e^{-t}+(e^{-t})^2=-e^{-t}+e^{-2t}=y.$$

x is positive because e^{-t} is positive.

(d) the equilibrium solution is a sink so all solutions tend to the origin as $t\to\infty$.

(e) see graph.