Maths 260 Lecture 5

Topic for today

More on Euler's method

Improved Euler's method

4th-order Runge-Kutta method

Reading for this lecture

BDH Sections 1.4, 7.1

Suggested Exercises

BDH Section 1.4: 1, 7; Section 7.1: 6

Reading for next lecture

BDH Sections 7.2, 7.3, 7.4

Today's handout

Lecture 5 notes

Section 1.4 continued: Euler's method

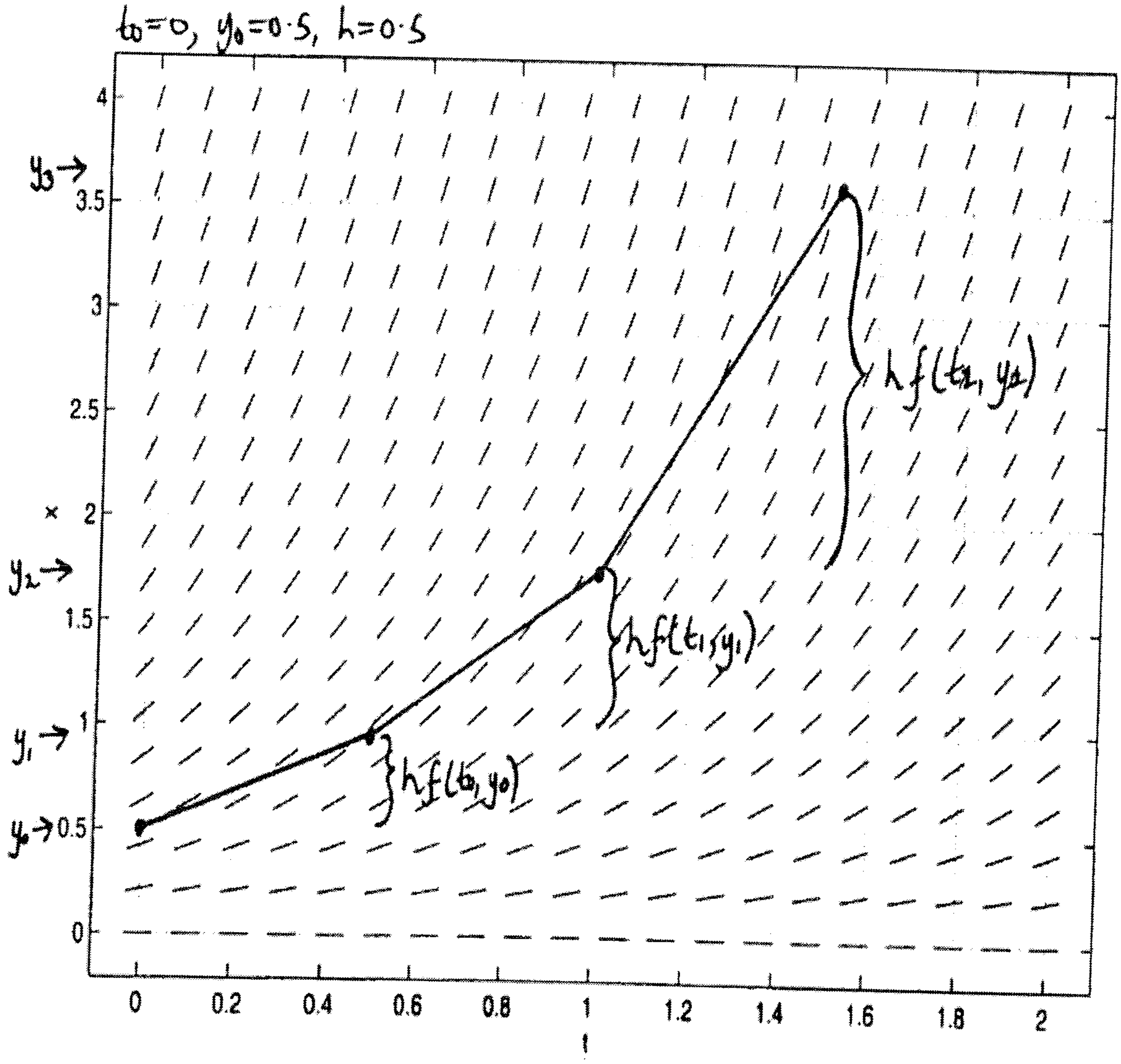
Recall from last lecture: Main idea of Euler's method

To approximate the solution to the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

start at (t_0, y_0) and take small steps, with the direction of each step being the direction of the slope field at the start of that step.

The following picture illustrates the relationship between the slope field and the numerical solution obtained from Euler's method.



In the next example we can solve the IVP exactly and hence check the accuracy of Euler's method for various choices of step size.

Example For the IVP

$$\frac{dy}{dt} = yt, \quad y(0) = 1$$

calculate an approximation to $y(0.4)$ using Euler's method with (i) $h = 0.2$, and (ii) $h = 0.1$. Calculate the error in each approximation.

Solution:

$y_0 = 1$

$h = 0.2, \quad t_0 = 0, \ t_1 = 0.2, \ t_2 = 0.4$

$y_1 = y_0 + h f(t_0, y_0)$

$= 1 + 0.2 (1.0)$ ← times

$= 1$

$y_2 = y_1 + h f(t_1, y_1)$

$= 1 + 0.2 (0.2 . 1)$

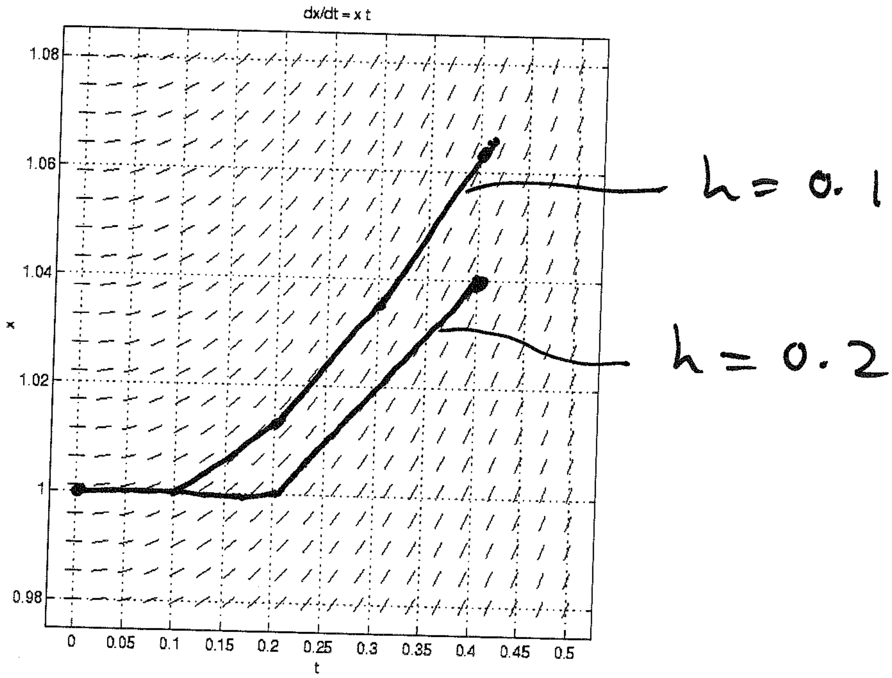
$= 1.04$

$h = 0.2$

n	t_n	y_n	$f(t_n, y_n)$	$y_n + hf(t_n, y_n)$
0	0.0	1.0	0.0	1.0
1	0.2	1.0	0.2	1.04
2	0.4	1.04		

$h = 0.1$

n	t_n	y_n	$f(t_n, y_n)$	$y_n + hf(t_n, y_n)$
0	0.0	1.0	0.0	1.0
1	0.1	1.0	0.1	1.01
2	0.2	1.01	0.202	1.0302
3	0.3	1.0302	0.30906	1.061106
4	0.4	1.0611		



To calculate the error in the approximation, we need to compare with the actual solution.

Exercise: Show that $y(t) = e^{t^2/2}$ solves the IVP.

Using the explicit solution, we get $y(0.4) = e^{(0.16)/2} \approx 1.0833$.

Error in the first approximation (with h=0.2) is $|1.0833 - 1.04| = 0.0433$

Error in the second approximation (with h=0.1) is $|1.0833 - 1.0611| = 0.0222$

Note that the error was approximately halved by halving the step size (but twice as many steps/calculations were done to obtain this improvement in accuracy).

When using Euler's method to get an approximate solution to an IVP, picking a smaller step size will usually give a more accurate approximation - but will involve more work. We return to this idea in the next lecture.

Section 1.5 Improving Euler's Method

Most elementary numerical methods, such as Euler's Method, can be understood in terms of approximating derivatives.

For small h we have

$$\frac{y(t_{n+1}) - y(t_n)}{h} \approx \frac{dy}{dt} = f(t, y)$$

So

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + \epsilon_n$$

where ϵ_n is the error made in the approximation.

Euler's Method approximates this formula by dropping ϵ_n from the equation above so that the Euler estimate at t_{n+1} is

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n))$$

Geometrically, Euler's method amounts to following a tangent line, instead of the (unknown) solution curve, from y_n to the value we accept for y_{n+1}.

The direction of each step is determined by slope at beginning of step.

Since the slope of the actual solution curve varies throughout the interval from t_n to t_{n+1}, the value of y_{n+1} calculated by Euler's method generally does not agree with the value on the solution curve.

We can obtain a more accurate method by adjusting the direction of the step according to the slope field seen along an Euler step.

Improved Euler's method (IE)

To take one step of length h with Improved Euler's method:

1. Take an ordinary Euler step of length h. Calculate the slope at the end of this step.
2. Go back to the beginning of the step, take a step of length h with slope being the average of the slope at the beginning of the step and the slope calculated in (1).

The formulas for this method are

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2)$$

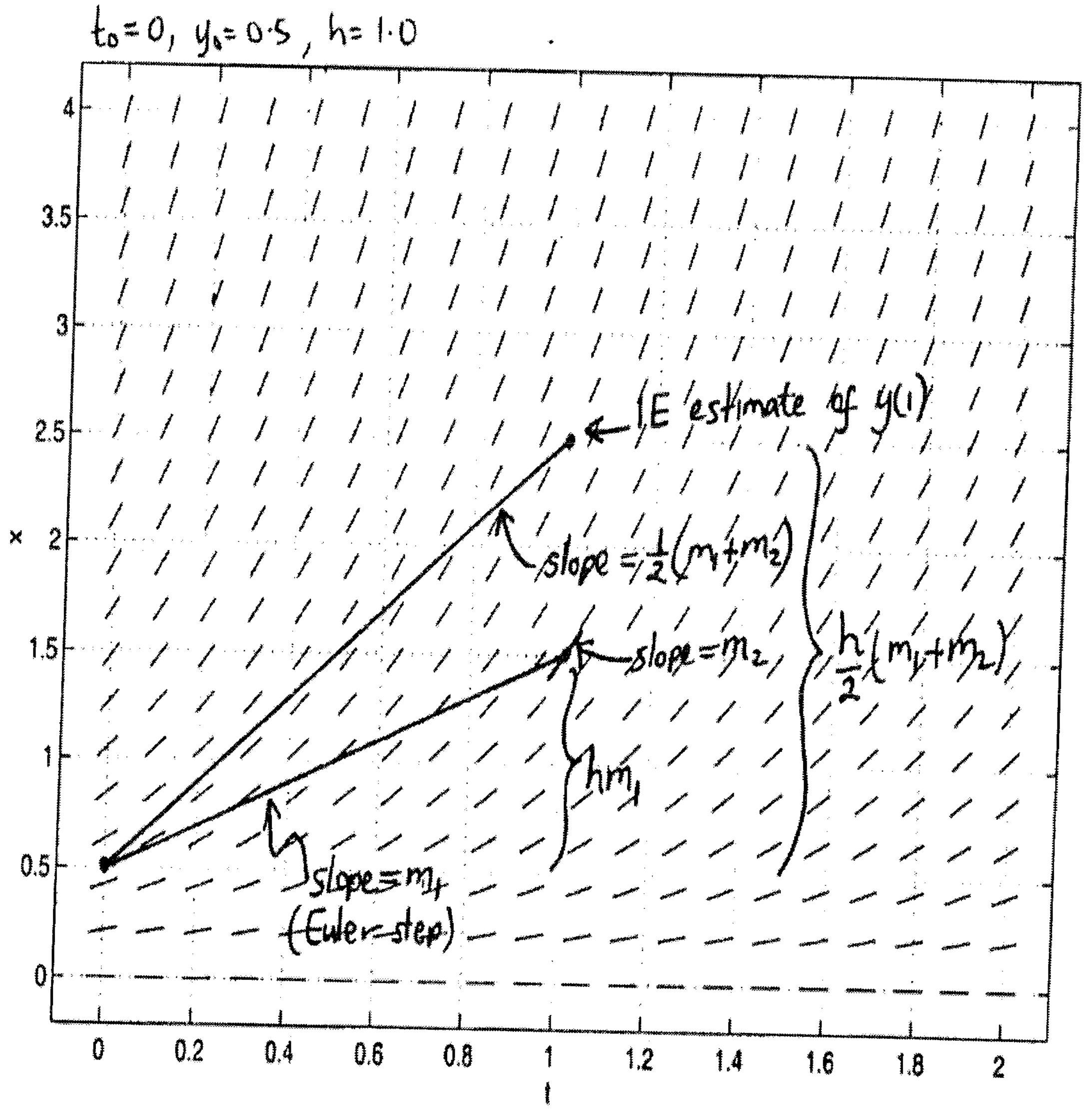
where

$$m_1 = f(t_n, y_n),$$
$$m_2 = f(t_{n+1}, y_n + h(f(t_n, y_n)))$$

slope at start (Euler slope)

slope at end (calculated using Eulers method)

The following picture illustrates the relationship between the slope field and the numerical solution obtained with IE method.



Example: Use $h = 0.5$ in the IE method to calculate an approximation to the solution of the IVP

$$\frac{dy}{dt} = -2ty^2, \quad y(0) = 1$$

at $t = 1.0$.

$$t_0 = 0, \quad t_1 = 0.5, \quad t_2 = 1.0$$

$$h = 0.5, \quad y_0 = 1$$

Step 1

$$m_1 = f(t_0, y_0) = -2 \cdot 0 \cdot 1^2 = 0$$

$$m_2 = f(t_1, y_0 + h \underbrace{f(t_0, y_0)}_{= m_1})$$

$$= f(0.5, 1 + 0)$$

$$= -2 \cdot 0.5 \cdot 1^2 = -1$$

$$y(0.5) \approx y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$$

$$= 1 + 0.25(0 + -1)$$

$$= 0.75$$

Step 2

$$m_1 = f(t_1, y_1) = -2(0.5)(0.75)^2$$

$$= -0.5625$$

$$m_2 = f(t_2, y_1 + h m_1)$$

$$= f(1,\ 0.75 + 0.5 m_1)$$

$$= \sout{-2.127}\ (-0.4395)$$

$$y_2 = y_1 + \frac{h}{2}(m_1 + m_2)$$

$$= 0.75 + 0.25(m_1 + m_2)$$

$$= \sout{0.0776}\ 0.4995$$

Using *numerical* from MATLAB, we can see how changing the step size in the IE method improves the solution.

Output from *numerical* for the above IVP, finding y value at $t = 0.5$:

No. of Steps	$y\,(1)$
1	0.7500000
2	0.7500000
4	0.7969455
8	0.7999361
16	0.8000542

(Compare with actual value: $y(0.5) = 0.8$).

We notice that accuracy is improved when a smaller step size is used.

4th-order Runge-Kutta method (RK4)

This is the most commonly used fixed-step size numerical method for IVPs.

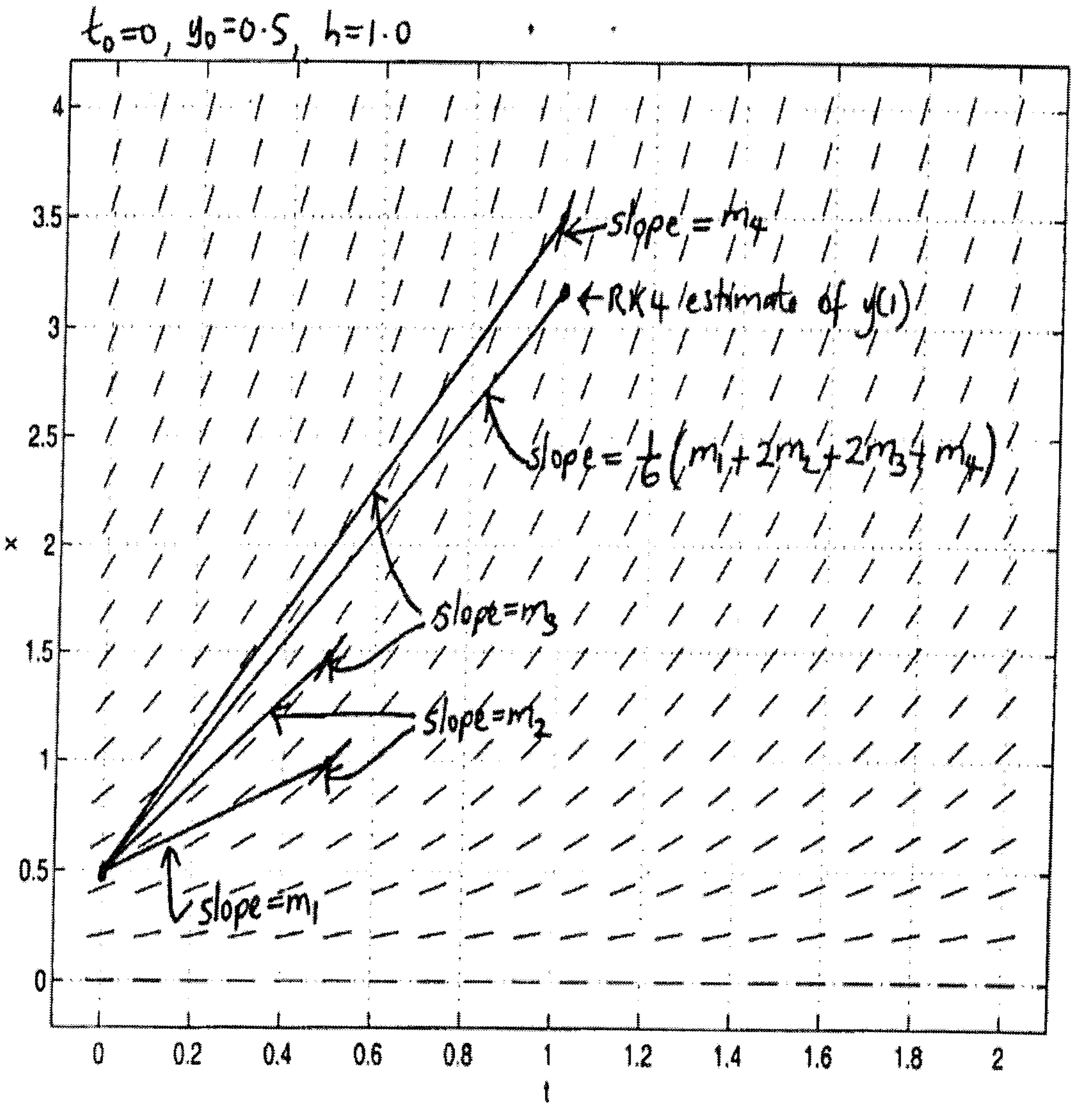
This method evaluates the slope $f(t, y)$ four times within each step. Starting at (t_n, y_n) we calculate (t_{n+1}, y_{n+1}) as follows:

$$
\begin{aligned}
t_{n+1} &= t_n + h \\
m_1 &= f(t_n, y_n) \\
m_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} m_1) \\
m_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} m_2) \\
m_4 &= f(t_n + h, y_n + h m_3)
\end{aligned}
$$

and now take

$$
y_{n+1} = y_n + \frac{h}{6} (m_1 + 2m_2 + 2m_3 + m_4)
$$

The following picture illustrates the relationship between the slope field and the numerical solution obtained with RK4 method.



Example: Use $h = 0.5$ and one step of RK4-method to calculate an approximation to the solution of the IVP

$$\frac{dy}{dt} = -2ty^2, \quad y(0) = 1$$

$t_0 = 0, \quad y_0 = 1, \quad h = 0.5$

$m_1 = f(t_0, y_0) = -2(0)(1)^2 = 0$

$m_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} m_1\right)$

$= f(0.25, 1) = -2(0.25)(1)^2$

$= -0.5$

$m_3 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} m_2\right)$

$= f(0.25, 1 + (0.25)(-0.5))$

$= -0.3828$ [illegible]

$m_4 = f(t_0 + h, y_0 + h m_3)$

$= f(0.5, 1 + 0.5(-0.3828))$

$= -0.6538$

~~$m_3 = -2(0.25)(1+(0.25)(-0.5))^2$~~

$=$

$$y(0.5) \simeq y_1 = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$= 0.7984$$

Important ideas from today

Numerical methods approximate solutions to IVPs

Euler's method uses the slope at the beginning of each step.

Better methods adjust the direction of each step according to the slope field seen along an Euler step.

The error in a numerical approximation generally reduces if the step size is decreased.

Maths 260 Lecture 6

Topics for today

Order of numerical methods
Efficiency of numerical methods
Comparison of methods seen so far

Reading for this lecture

BDH Section 7.4

Suggested exercise

BDH Section 7.3: 6

Reading for next lecture

BDH Section 2.1

Today's handouts

Lecture 6 notes
Tutorial 2 Questions

Order of Numerical Methods

The **order** of a numerical method measures the change in error of a numerical solution as step size is decreased.

With Euler's method we find that the error is approximately halved if the step size is halved (at least if h is sufficiently small).

For example, Euler's method gives us the following results for the IVP

$$\frac{dy}{dt} = y, \; y(0) = 1.$$

$y = e^t$

$(y(1) = 2.71..)$

No. of Steps	$y(1)$	error
1	2.000000	0.718
2	2.250000	0.468
4	2.441406	0.277
8	2.565784	0.152
16	2.637928	0.0804
32	2.676990	0.0413
64	2.687345	0.0209
128	2.707739	0.0105
256	2.712992	0.00529
512	2.715632	0.00265

Looking at the same IVP with Improved Euler yields:

$y(1) = 2.71..$

$0.21 \times (0.5)^2$

No. of Steps	$y(1)$	error
$h=1$ 1	2.500000	0.218
$h=0.5$ 2	2.640625	0.0777
$h=0.25$ 4	2.694856	0.0234
8	2.711841	0.00644
16	2.716594	0.00169
32	2.717850	0.000432
64	2.718173	0.000109
128	2.718254	0.0000274
256	2.718275	0.00000689
512	2.718280	0.00000173

decreasing by 4

The same IVP with RK4 yields:

No. of Steps	$y(1)$	error
1	2.708333	0.00994
2	2.717346	0.000936
4	2.718210	0.0000719
8	2.718277	0.00000498
16	2.718282	0.000000328
32	2.718282	0.0000000215

decreasing by 16

We see that:

1. for a fixed step size we get a smaller error with IE and RK4 than with Euler's method, i.e., IE and RK4 are more accurate methods than Euler's method for this problem;
2. (more importantly) for IE and RK4 the error decreases faster as the step size is reduced than with Euler's method.

Define $E(h)$ to be the error in the approximate solution obtained when we solve an IVP using a particular numerical method with step size h.

If $|E(h)| \approx kh^p$ as $h \to 0$ then p is called the **order of the numerical method**.

Here, k is a constant depending on the IVP and the method. Error for

Eulers	kh
I.E	kh^2
R-K	kh^4

Estimating the order of a numerical method

For a particular IVP and a particular method of order p,

$$\lim_{h\to 0}\frac{|E(2h)|}{|E(h)|}\approx \lim_{h\to 0}\frac{k(2h)^p}{kh^p}=2^p$$

Taking logs,

$$\ln\left|\frac{E(2h)}{E(h)}\right|\approx p\ln 2$$

or $\quad$ ($\ln\left|\frac{a}{b}\right| = \ln|a| - \ln|b|$)

$$p\approx\frac{\ln|E(2h)|-\ln|E(h)|}{\ln 2}$$

(for small h).

For a particular choice of h, the quantity

$$q=\frac{\ln|E(2h)|-\ln|E(h)|}{\ln 2}$$

is called the **effective order of the method** at step size h. We expect $q\to p$ as $h\to 0$.

(p = exact order & q is an estimate)

Example: Consider the IVP

$$\frac{dy}{dt} = y, \quad y(0) = 1$$

Estimates for $y(1)$ calculated with various step sizes were given earlier.

Use these estimates to calculate the effective order of Euler's method for this IVP at $h = 0.125$. Repeat for IE and RK4.

$$E(h) = E(0.125) = 0.00644$$

$$E(2h) = E(0.25) = 0.0234$$

$$q = \frac{\ln|E(2h)| - \ln|E(h)|}{\ln 2}$$

$$= \frac{\ln(0.0234) - \ln(0.00644)}{\ln(2)}$$

$$= 1.86 \quad (\approx 2)$$

(if we choose h smaller it would be closer to 2)

It can be proved that

- Euler's method is of order 1
- Improved Euler is of order 2
- Runge-Kutta 4 is of order 4

Efficiency of numerical methods

Higher order methods take more work per step than Euler's method. However, they usually require fewer steps and less total work to obtain an accurate answer.

The most efficient method for a particular problem is the one which gives the desired accuracy with the least amount of work.

In estimating the amount of work required to use a certain method, we only count the number of evaluations of f. In comparison, additions and multiplications required are negligible.

Euler

Example: For the IVP discussed at the start of lecture which of Euler, IE and RK4 is most efficient if an accuracy of 1% in the solution at $t = 1$ is required?

(Exact soln 2.7182

ie error less that 0.027)

Euler	64	steps
I.E.	4	steps
R-K	1	step

—— // ——

Work is one / step Euler
two / step I.E
four / step R-K

Work for 1% accuracy is

Euler	64	calculations
I.E.	8	calculations
R-K	4	calculations

The Dormand-Prince method

The Matlab functions *dfield* and *pplane* use the Dormand-Prince numerical method. We do not study this method in detail, but note that it incorporates three improvements over RK4:

1. It is a 5th order method.
2. A variable step size is used. The algorithm calculates the step size to be used by estimating the error in each step. A large error estimate will cause a smaller step size to be used.
3. Some fitting (with splines) is performed to give smoother solutions.

Comparison of Methods seen so far

We have seen examples of three important types of methods for getting information about solutions to DEs.

1. **Qualitative methods** (e.g. slope fields) are useful for understanding qualitative properties of solutions (e.g. long term behaviour) but do not give exact values of solutions at particular times.

2. **Analytic methods** (e.g. solving separable equations) give a formula for a solution of a DE. These methods work in some important special cases but do not work in most cases.

3. **Numerical methods** (e.g. Euler's method) give approximate quantitative information about solutions. We can automate these methods (i.e. use a computer). These methods can be misleading, and give information about only one solution at a time.

It is usually possible to use more than one method for any problem - the trick is picking the most appropriate method(s).

We will learn more about each class of methods in the rest of the course.