Example: Find the general solution to

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = t$$

Describe the long term behaviour of solutions.

First solve homogeneous eqn

$$y_{gh}'' + 4y_{gh}' + 3y_{gh} = 0$$

char. poly $\lambda^2 + 4\lambda + 3 = 0$

$$\lambda = -3, -1$$

$$y_{gh} = c_1 e^{-3t} + c_2 e^{-t}$$

Now find y_p

guess $y_p = At + B$

substitute into

$$y_p'' + 4y_p' + 3y_p = t$$

$$\Rightarrow \quad 4A + 3(At+B) = t$$

$$\Rightarrow \quad A = 1/3, \quad B = -4/9$$

$$y_p = 1/3\, t - 4/9$$

$$y = c_1 e^{-3t} + c_2 e^{-t} + 1/3\, t - 4/9$$

at $t \to \infty$ $y \to 1/3\, t - 4/9$

(ie. after a long time the soln
will be $y(t) = \frac{1}{3}t - \frac{4}{9}$
no matter what initial y & y' are)

Example: Find the solution to the IVP

$$y'' - 2y' - 3y = te^{3t},$$
$$y(0) = 0, \quad y'(0) = 1$$

Find y_h solves

$$y_h'' - 2y_h' - 3y_h = 0$$

$$\Rightarrow y_h = c_1 e^{-t} + c_2 e^{3t}$$

Now find y_p

Guess $y_p = Ate^{3t} + Be^{3t}$

(but e^{3t} is a soln to homogeneous eqn)

Have to multiply by t

$$y_p = At^2e^{3t} + Bte^{3t}$$
$$y_p' = 3At^2e^{3t} + (2A+3B)te^{3t} + Be^{3t}$$
$$y_p'' = \cancel{8}9At^2e^{3t} + (12A+9B)te^{3t} + (2A+6B)e^{3t}$$

substitute into

$$y_p'' \cancel{+} - 2y_p' - 3y_p = te^{3t}$$

$$\Rightarrow \overbrace{9At^2e^{3t} + (12A+9B)te^{3t} + (2A+6B)e^{3t}}^{y_p''}$$

$$2y' \rightarrow \quad - 6At^2e^{3t} - (4A+6B)te^{3t} - 2Be^{3t}$$

$$- 3At^2e^{3t} - 3Bte^{3t} = te^{3t}$$

$$(+9A - 6A - 3A)t^2e^{3t}$$

$$+ (12A + 9B - 4A - 6B - 3B)te^{3t}$$

$$+ (2A + 6B - 2B)e^{3t} = te^{3t}$$

$$\Rightarrow (8A)te^{3t} + (2A+4B) = te^{3t}$$

$$\left.\begin{array}{l} 8A = 1 \\ 2A + 4B = 0 \end{array}\right\} \begin{array}{l} A = 1/8 \\ B = -1/16 \end{array}$$

$$y_p = 1/8\, t^2e^{3t} - 1/16\, te^{3t}$$

$$y = C_1e^{-t} + C_2e^{3t} + 1/8\, t^2e^{3t} - 1/16\, te^{3t}$$

Maths 260 Lecture 32

Topics for today

More nonhomogeneous higher order DEs

Forcing and resonance in the harmonic oscillator

Reading for this lecture

BDH Sections 4.3, 4.4

Suggested exercises

BDH Section 4.3; 3, 7, 10, Section 4.4; 2,

Reading for next lecture

BDH Section 6.1

Today's handouts

Lecture 32 notes

Recall from last lecture :

<u>Summary of method of solution for nonhomogeneous equations of the form</u>

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

1. Find the general solution to the related homogeneous equation.
2. Find <u>one</u> solution to the non-homogeneous equation.
3. Add answers to (1) and (2) to get the general solution to the nonhomogeneous equation.
4. If trying to solve an IVP, use the initial conditions to determine constants in the general solution.

To find a particular solution to the nonhomogeneous DE where f(t) is

(i) a constant, or

(ii) t^n for n a positive integer, or

(iii) $e^{\lambda t}$ for real $\lambda \neq 0$, or

(iv) $\sin bt$ or $\cos bt$, b constant, or

(v) a finite product of terms like (i)-(iv),

take the following steps:

1. Form the UC set consisting of f and all linearly independent functions obtained by repeated differentiation of f.
2. If any of the functions in the UC set is also a solution to the homogeneous DE, multiply all functions in the set by t^k, where k is the smallest integer so that the modified UC set does not contain any solutions to the homogeneous DE.

3. Find a particular solution to the DE by taking a linear combination of all the functions in the (possibly modified) UC set. Determine the unknown constants by substituting this linear combination into the DE.

We can modify this method to solve DEs for which the forcing function is a finite sum of terms.

For example, to find a solution to

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f_1(t) + f_2(t)$$

we use linearity, i.e., first find y_1 that solves

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f_1(t)$$

and then find y_2 that solves

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f_2(t).$$

Then

$$y = y_1 + y_2$$

solves the DE with forcing term

$$f(t) = f_1(t) + f_2(t)$$

Example: Find the general solution to

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 2\sin t - e^{2t}$$

Find y_h

$$y_h'' - 3y_h' + 2y_h = 0$$

$$y_h = c_1 e^t + c_2 e^{2t}$$

Find y_p for

$$y_p'' - 3y_p' + 2y_p = 2\sin t \quad (*)$$

guess $y_p = A\sin t + B\cos t$

$$y_p' = +A\cos t - B\sin t$$

$$y_p'' = -A\sin t - B\cos t$$

substitute into $(*)$

$$-A\sin t - B\cos t - 3A\cos t + 3B\sin t$$

$$\cancel{+2A\cos t + 2B\sin}$$

$$+2A\sin t + 2B\cos t = 2\sin t$$

$$(-A + 3B + 2A)\sin t$$

$$+(-B - 3A + 2B)\cos t = 2\sin t$$

$$\Rightarrow \quad \left.\begin{aligned} \cancel{3A}\ 3B + A &= 2 \\ -3A + B &= 0 \end{aligned}\right\} \quad \begin{aligned} B &= 3/5 \\ A &= \tfrac{1}{5} \end{aligned}$$

$$y_p = 1/5 \sin t + 3/5 \cos t$$

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Now solve

$$y_p'' \cancel{\$} - 3y_p' + 2y_p = -e^{2t} \quad (**)$$

guess $y_p = Ae^{2t}$

but e^{2t} solves homogeneous eqn.

$$y_p = Ate^{2t}$$
$$y_p' = 2Ate^{2t} + Ae^{2t}$$
$$y_p'' = 4Ate^{2t} + 4Ae^{2t}$$

& substitute into $(**)$

$$\overbrace{4Ate^{2t} + 4Ae^{2t}}^{y_p''} \overbrace{- 6Ate^{2t} - 3Ae^{2t}}^{-3y_p'} + \underbrace{2Ate^{2t}}_{2y_p} = -e^{2t}$$

$$(4A - 6A + 2A)te^{2t} + (4A - 3A)e^{2t} = -e^{2t}$$

$$\Rightarrow A = -1$$

$$y_p = -te^{2t}$$

Soln to $y'' - 3y' + 2y = 2\sin t - e^{2t}$

is

$$y = c_1 e^{t} + c_2 e^{2t}$$
$$+ 1/5 \sin t + 3/5 \cos t$$
$$- t e^{2t}$$

Example: Find the general solution to

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = t + 2e^t$$

Find soln for

$$y_h''' + 2y_h'' + 5y_h' = 0$$

char polynomial $\lambda^3 + 2\lambda^2 + 5\lambda = 0$

$$\Rightarrow \quad \lambda = 0, \ -1 \pm 2i$$

$$y = c_1 + c_2 e^{-t}\cos 2t + c_3 e^{-t}\sin 2t$$

Now solve

$$y_p''' + 2y_p'' + 5y_p' = t \quad (*)$$

guess $y_p = At + B$

but B is a soln of homogeneous eqn.

modify

$$y_p = At^2 + Bt$$

& substitute into $(*)$

$$2(2A) + 5(2At + B) = t$$

$$10At + 4A + 5B = t$$

$$A = 1/10 \quad , \quad B = -2/25$$

$$y_p = 1/10\, t^2 - 2/25\, t$$

Now find y_p for

$$y_p''' + 2y_p'' + 5y_p' = 2e^t \quad (**)$$

$$y_p = Ae^t$$

& substitute into $(**)$

$$Ae^t + 2Ae^t + 5Ae^t = 2e^t$$

$$A = 1/4$$

$$y_p = 1/4\, e^t$$

$$y = C_1 + C_2 e^{-t} \cos 2t + C_3 e^{-t} \sin 2t$$
$$+ 1/10\, t^2 - 2/25\, t + 1/4\, e^t$$

The forced harmonic oscillator

A forced higher order equation of special interest in applications is the periodically forced harmonic oscillator, i.e.,

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos \omega t$$

for constants $p \geq 0$, $q > 0$ and $\omega > 0$. We are interested in the longterm behaviour of solutions for various values of the damping coefficient p.

The characteristic polynomial is

$$\lambda^2 + p\lambda + q = 0$$

which has roots

$$\lambda = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

Thus, in the case $0 \leq p^2 < 4q$, the homogeneous equation has general solution

$$y_c = c_1 e^{-\frac{p}{2}t} \cos \alpha t + c_2 e^{-\frac{p}{2}t} \sin \alpha t$$

where

$$\alpha = \frac{\sqrt{4q - p^2}}{2}$$

Exercise : Find the general solution to the homogeneous problem when $p^2 \geq 4q$.

Exercise : Show that for all values of $p > 0$ the solution to the homogeneous equation tends to zero as $t \to \infty$.

We see that as $t \to \infty$, all solutions to the forced harmonic oscillator with nonzero damping behave the same, i.e., like the particular solution to the non-homogeneous problem.

To find the particular solution :

$$y_p'' + p y_p' + q y_p = \cos \omega t$$

substitute $y_p = A \cos \omega t + B \, \cancel{\cos}\, \sin \omega t$

(provided $\cos \omega t$ is not a soln of homogeneous eqn.

i.e. $\alpha \neq \omega$ and $p \neq 0$)

We can write

$$y_p = C \cos(\omega t + \theta)$$

(unknown's are C & θ not A & B)

We find

$$y_p = A\cos(wt + \theta)$$

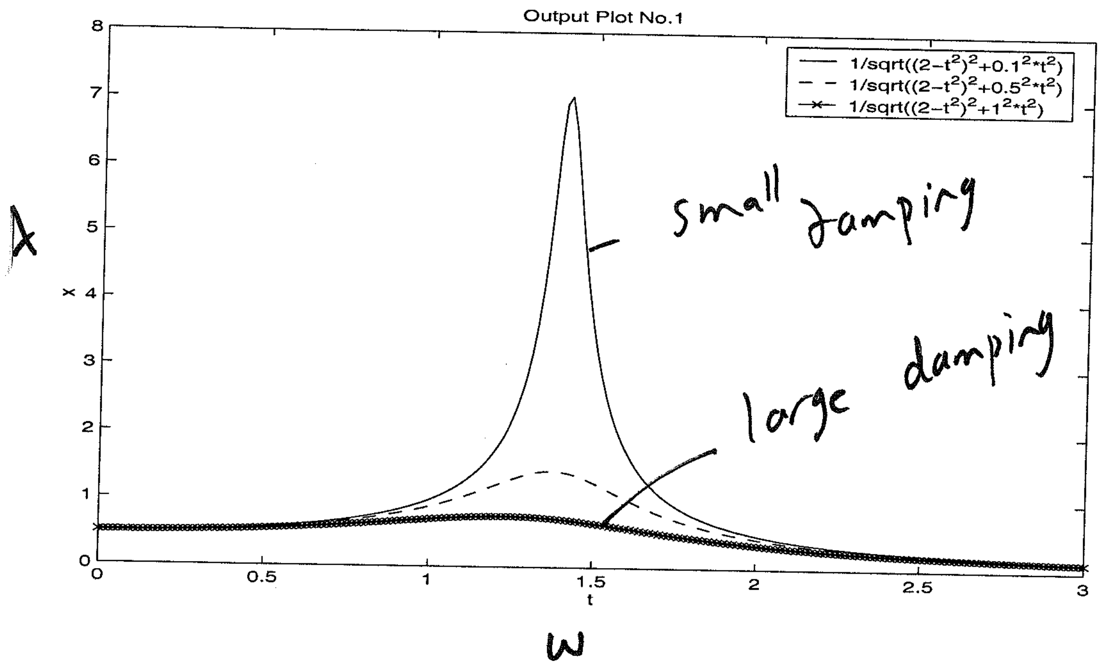
where

$$A = \sqrt{\frac{1}{(q - \omega^2)^2 + (\omega p)^2}},$$

$$\theta = \tan^{-1}\left(\frac{-\omega p}{q - \omega^2}\right)$$

<u>Qualitative behaviour of this solution</u> :

Amplitude of y_p as a function of ω in the case $q = 2$ and various p :



In the undamped case ($p = 0$), a particular solution is

$$y_p = \begin{cases} \frac{1}{q-\omega^2} \cos \omega t, & \omega^2 \neq q \\ \\ \frac{1}{2\omega} t \sin \omega t, & \omega^2 = q. \end{cases}$$

Exercise : Check this.

special case when we multiply by t

Qualitative behaviour of this solution :

Summary :

For the damped periodically forced harmonic oscillator

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos\omega t$$

solutions eventually become periodic with frequency the same as the forcing frequency, ω, and with amplitude depending on ω.

The amplitude of the long term solutions can be very large if the forcing frequency (ω), is close to the natural frequency of the unforced system ($\sqrt{q}$).

The smaller the damping (p), the closer the frequency of the long term solution is to the natural frequency, and the larger the amplitude of the long term solutions.