# Maths 260 Lecture 7

## Topic for today

Existence and uniqueness of solutions

## Reading for this lecture

BDH Section 1.5

## Suggested Exercises

BDH Section 1.5: 1, 3, 5, 7, 15

## Reading for next lecture

BDH Section 1.6, pp 74-80

## Today's handout

Lecture 7 notes

## §1.5 Existence and Uniqueness of solutions

In the theory and examples we have studied already we have been making two major assumptions: that the DEs we study really have solutions and that such solutions are unique.

On the whole we are safe in making these assumptions. Today we shall see why.

## Existence Theorem

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

If $f(t, y)$ is a continuous function of $t$ and of $y$ at $(t, y) = (t_0, y_0)$, then there is a constant $\epsilon > 0$, and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ solves the IVP

Note: The theorem guarantees a solution exists for a small interval in $t$, but says nothing about existence for all $t$.

do not need to understand!

"Fancy way of saying soln exist only for some time."

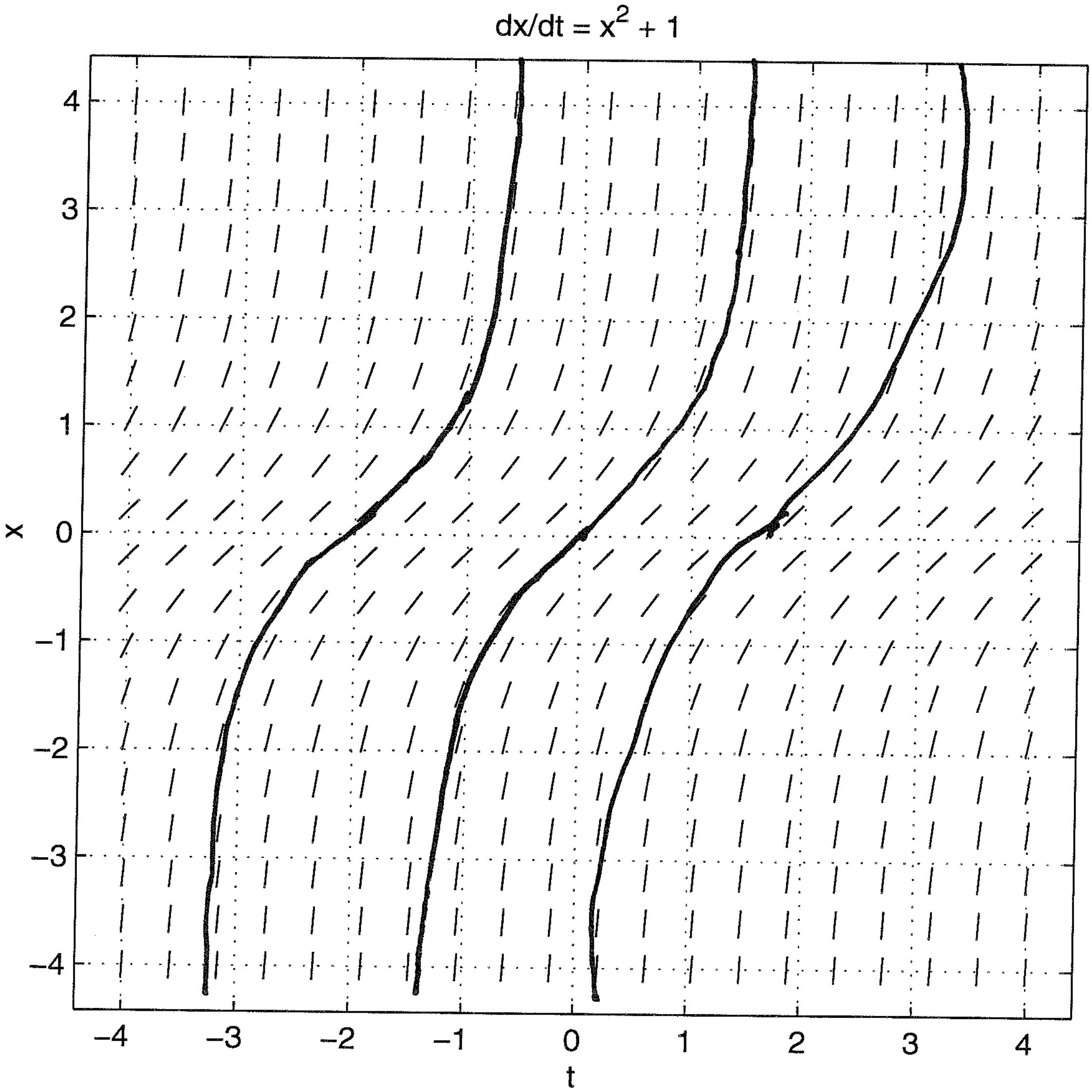
Example: Consider the IVP

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

$f(t,y) = 1+y^2$ is cts

Does the IVP have a solution? For what values of $t$ does the solution exist?

The slope field for the DE is:



Here $f(t, y) = 1 + y^2$ is a continuous function of $t$ and of $y$ for all $t, y$, so the Existence Theorem ensures a solution to the IVP exists for $-\epsilon < t < \epsilon$, for some $\epsilon$.

In fact, $y(t) = \tan(t)$ is a solution to IVP and is defined for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ but not for all t.

( Example shows that even for nice $f(t,y)$ soln goes to $\infty$ in finite time $\Rightarrow$ soln only exist in an interval $t \in [-\varepsilon, \varepsilon]$

## Uniqueness Theorem

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions of $t$ and of $y$ at $(t, y) = (t_0, y_0)$, then there is an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ is the unique solution to the IVP on this interval.

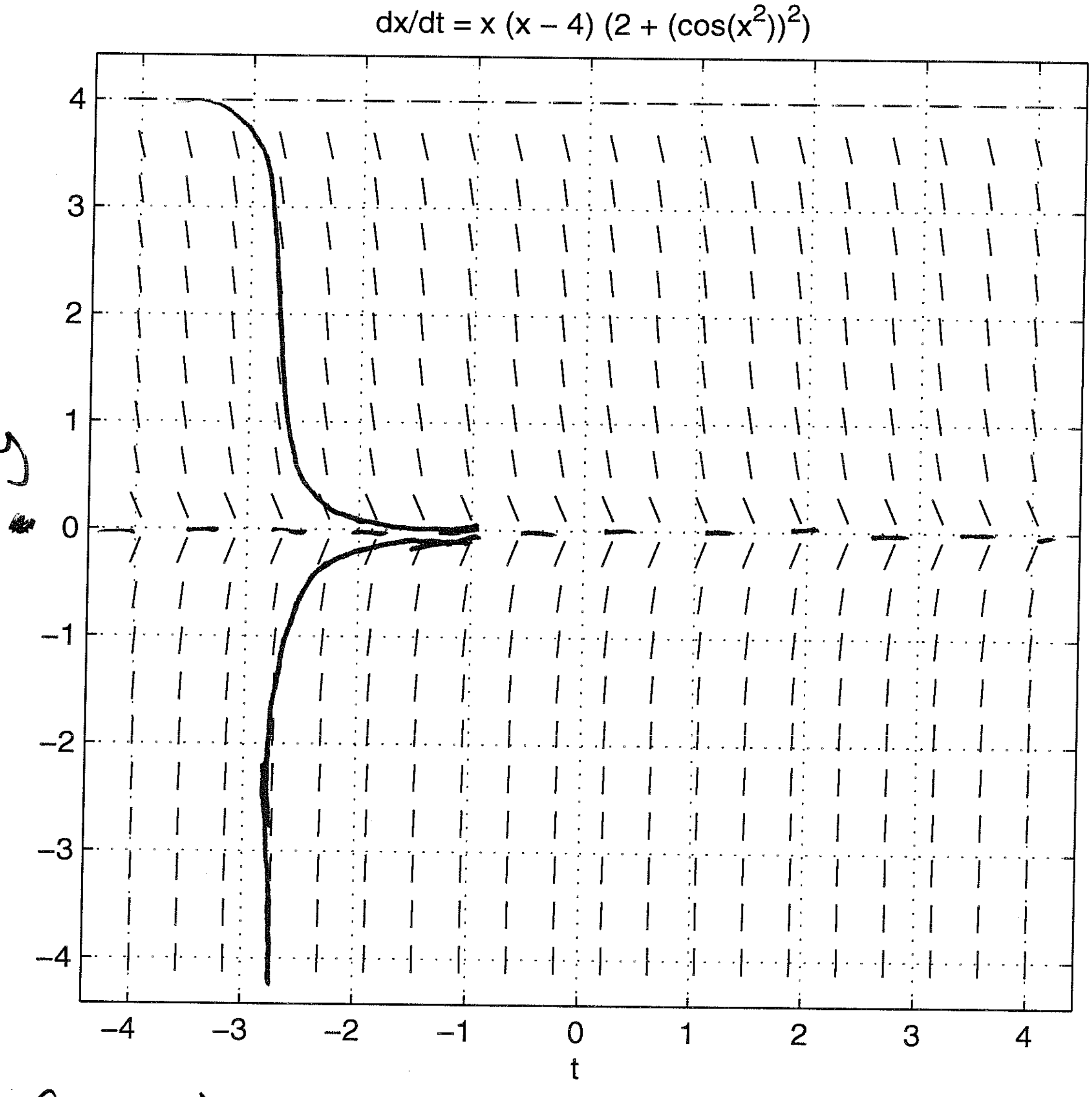
Note: The Uniqueness Theorem implies that different solutions can never cross or meet in $(t, y)$ plane.

In case $f$ cts but $\frac{\partial f}{\partial y}$ not cts then we have many solns.

## Examples where the Existence and Uniqueness Theorems are useful

1. $\dfrac{dy}{dt} = y(y-4)(2+\cos^2(y^2))$, $y(0)=1$ $= f(t,y)$

   What is the qualitative behaviour of solutions to the IVP?




$f(t,y)$ is cts and $\dfrac{\partial f}{\partial y}$ is cts at every point $\Rightarrow$ unique soln.

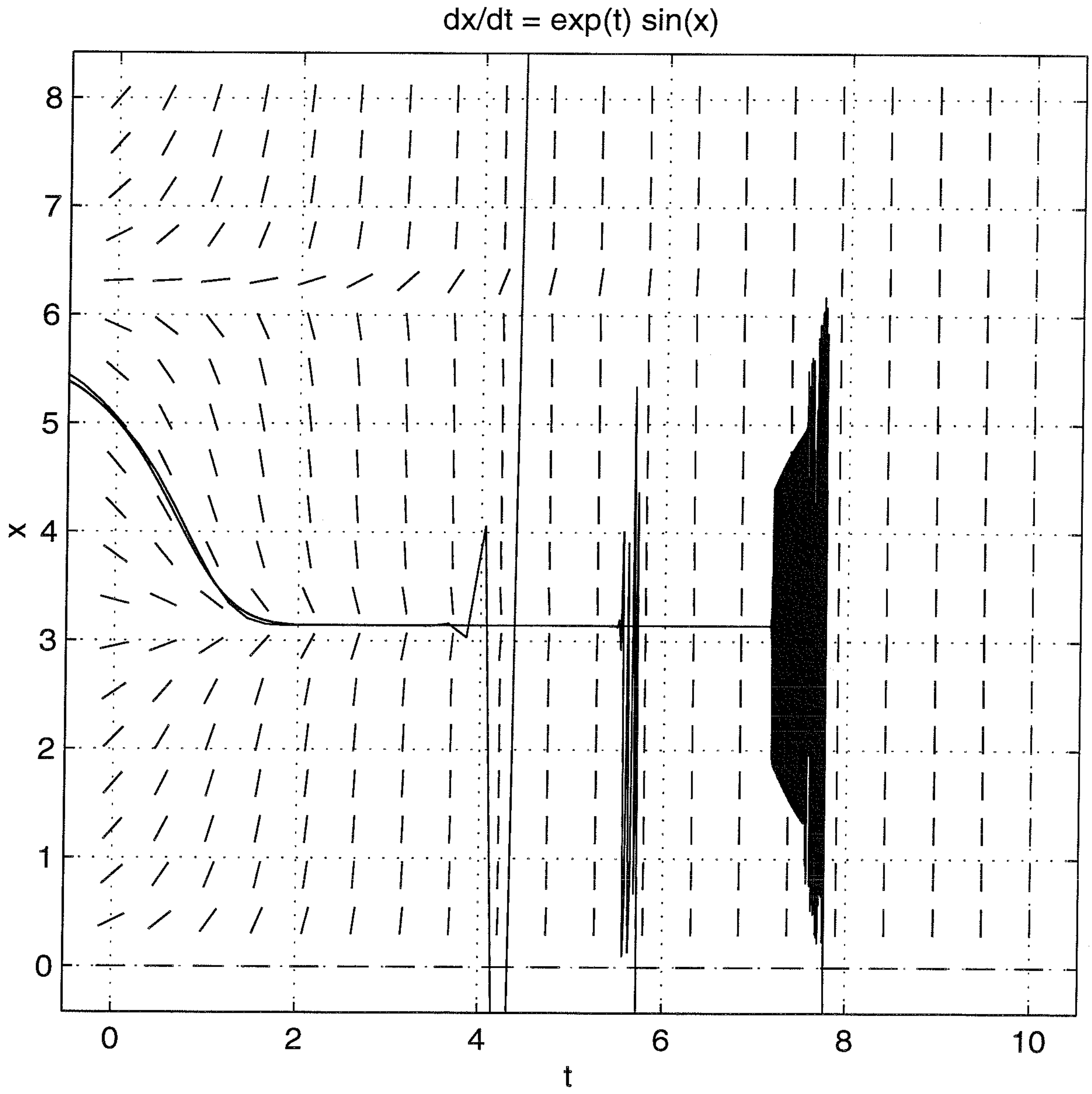
2. For the IVP

$$\frac{dy}{dt} \overset{= f(t,y)}{=} e^t \sin(y), \quad y(0) = 5$$

use the function *dfield* from Matlab and Euler's method with various step sizes to determine the behaviour of the solution to the DE.

$$\left.\begin{aligned} f(t,y) &= e^t \sin y \\ \frac{\partial f}{\partial y} &= e^t \cos y \end{aligned}\right\} \text{both cts}$$

$\Rightarrow$ we have a unique soln with initial value $y(0) = 5$ (and for all initial values)




(we know unique soln exists
but we cannot find it
numerically)

3. Given the IVP

$$\frac{dy}{dt} = ty^{\frac{1}{5}}, \quad y(t_0) = y_0$$

(a) Find a value of $t_0$ and a value of $y_0$ so that the IVP has a unique solution. Give a reason for your answer.

(b) Find a value of $t_0$ and a value of $y_0$ so that the IVP has more than one solution. For your choice of $t_0$ and $y_0$ write down two functions that satisfy the DE.

$f(t,y)$ is special

$f(t,y)$ is cts

$\frac{\partial f}{\partial y}(t,y) = \frac{1}{5} t\, y^{-4/5}$ is not cts at $y=0$ (zero)

i.e. at $y=0$ we have existence of a soln but not uniqueness.

(a) For all $t$ and all $y$ except $y = 0$ $f$ & $\frac{\partial f}{\partial y}$ are cts and we have unique soln.

i.e. $t = 903.8$, $y = 10.1$

(b) Any value of $t$ & $y = 0$ we do not have $\frac{\partial f}{\partial y}$ cts

so we do not have unique soln.

i.e. $y(0) = 0$ $(y_0 = 0, t_0 = 0)$

Solve $ty^{1/5} = \cancel{\partial f} \frac{dy}{dt}$

$$\int \frac{1}{y^{1/5}} dy = \int t dt$$

$$\int y^{-1/5} dy = \int t dt$$

$$= \frac{5}{4} y^{4/5} = \frac{t^2}{2} + c$$

$(y(0) = 0) \Rightarrow \cancel{t} \; c = 0$

$$y = \left(\frac{4}{5} \frac{t^2}{2}\right)^{5/4}$$

Also have a soln

$$y = 0 \quad ( y(t) = 0 )$$

## Important ideas from today

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

If $f$ is 'nice', a solution to the IVP exists, at least for $t$ near $t_0$.
Also, if $f$ and $\frac{\partial f}{\partial y}$ are 'nice', the solution to the IVP is unique. This implies that solution curves won't cross or touch in $(t, y)$ space.

# Maths 260 Lecture 8

## Topic for today

The phase line

## Reading for this lecture

BDH Section 1.6, pp 76-85

## Suggested Exercises

BDH Section 1.6: 23, 25, 27, 29

## Reading for next lecture

BDH Section 1.6, pp 81-88

## Today's handout

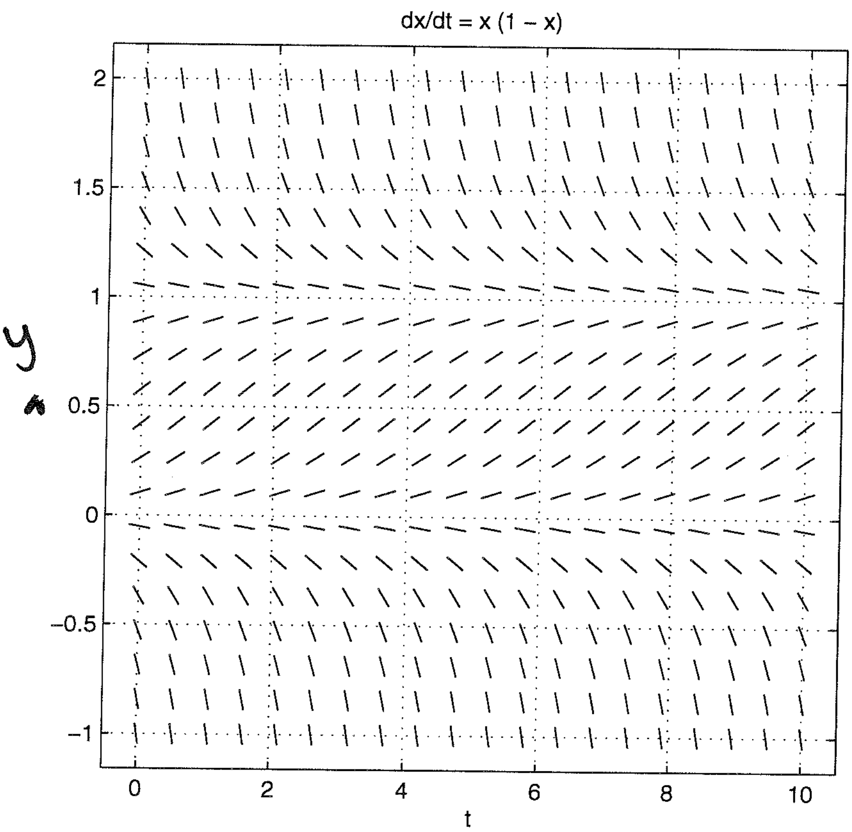
Lecture 8 notes

## §1.6 The Phase line

Consider the DE $\dfrac{dy}{dt} = f(y)$.

does not involve $t$ (called autonomous)

Recall that the slope field corresponding to an autonomous differential equation has a special form - slope marks are parallel along horizontal lines.




constant slope

There is clearly some redundancy in slope field information. We can replace the slope field by a phase line, which summarises the information in the slope field.

To draw a phase line for $\frac{dy}{dt} = f(y)$

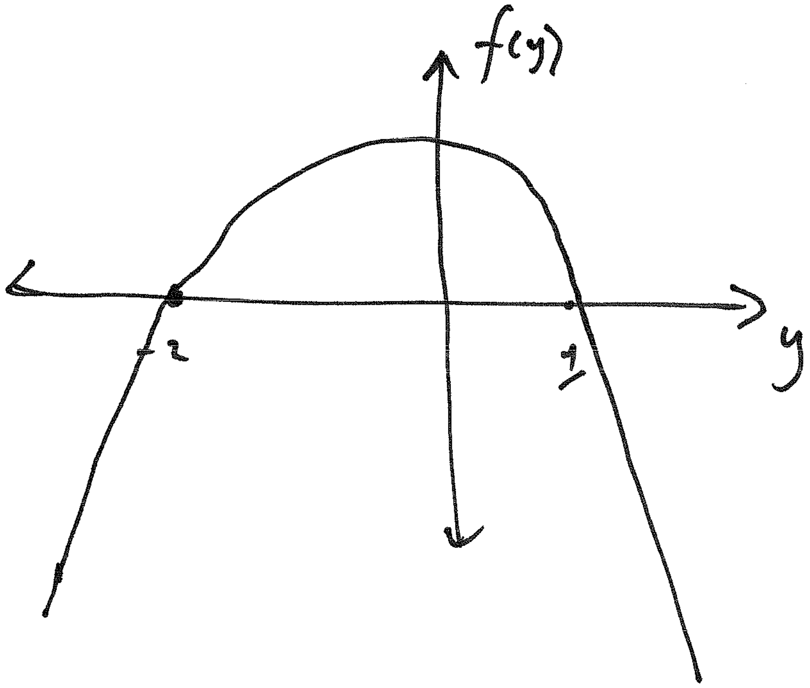
1. Draw $y$-line.
2. Find equilibrium solutions of the DE and mark them on the line.
3. Find intervals of $y$ for which $f(y) > 0$ (solutions started at such y values will increase as t increases). Draw upward pointing arrows on the line in these intervals.
4. Find intervals of $y$ for which $f(y) < 0$ (solutions started at such y values will decrease as t increases). Draw downward pointing arrows on the line in these intervals.

# Examples

1. For the DE $\dfrac{dy}{dt} = (y+2)(1-y) = f(y)$ sketch the phase line. Describe the longterm behaviour of solutions.

Recall $f(y) > 0 \Rightarrow y$ increases in time

$f(y) < 0 \Rightarrow y$ decreases in time




If

| | |
|---|---|
| $y < -2$ | $f(y) < 0$ |
| $y = -2$ | $f(y) = 0$ |
| $-2 < y < 1$ | $f(y) > 0$ |
| $y = 1$ | $f(y) = 0$ |
| $y > 1$ | $f(y) < 0$ |



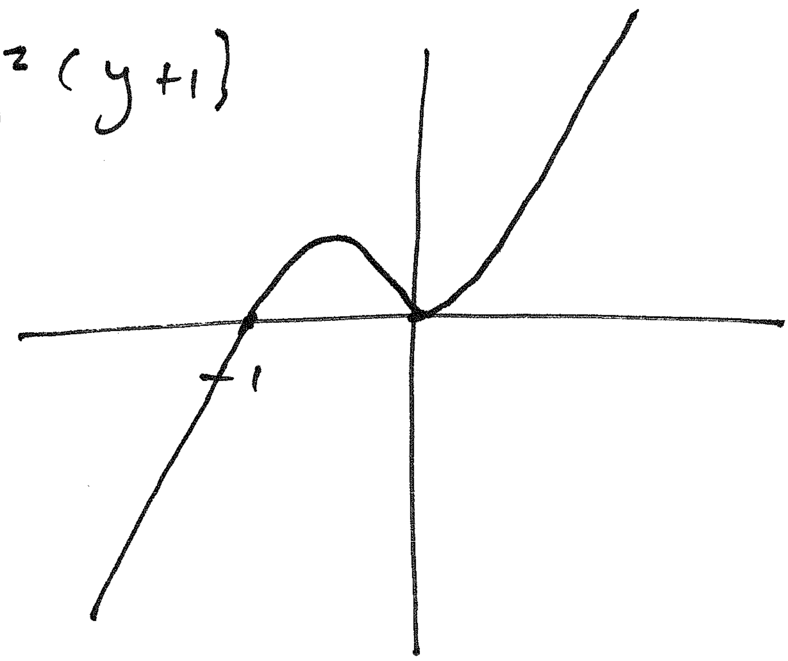
( Equilibrium point $\frac{dy}{dt} = 0 \Rightarrow f(y) = 0$ )

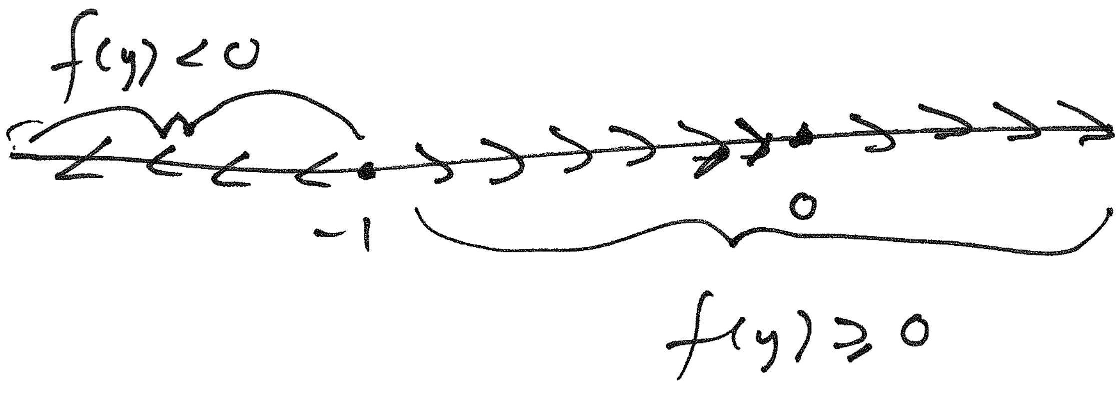
1 = equilibrium point ($f(y) = 0$)

$-2$ = equil. point ($f(y) = 0$)

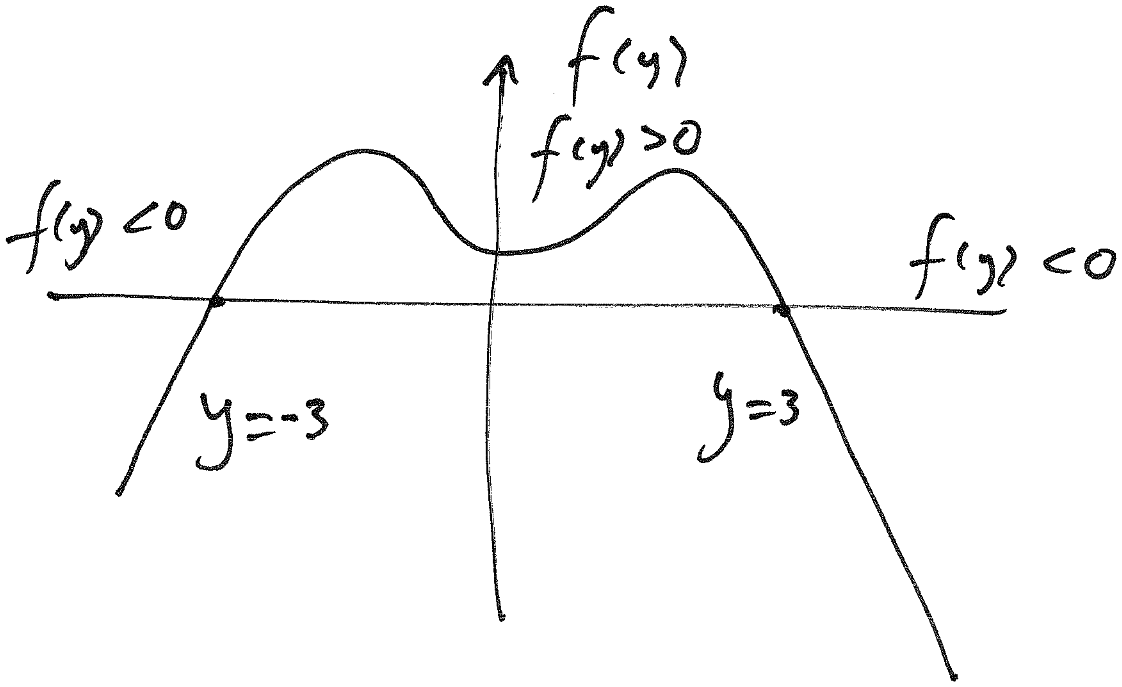
2. For the DE $\dfrac{dy}{dt} = y^2(y+1)$ sketch the phase line. Describe the longterm behaviour of solutions.

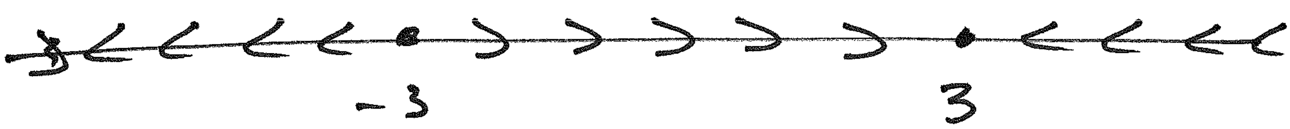
$f(y) = y^2(y+1)$



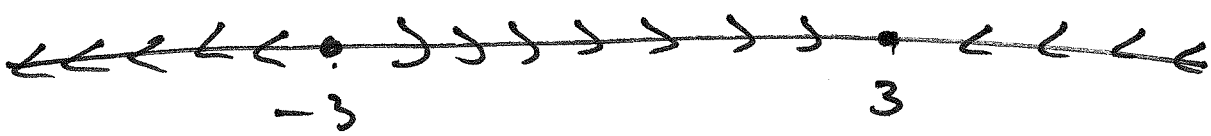
3. For the DE $\dfrac{dy}{dt} = f(y)$ where $f(y)$ has the graph shown below, sketch the phase line and describe the longterm behaviour of solutions.

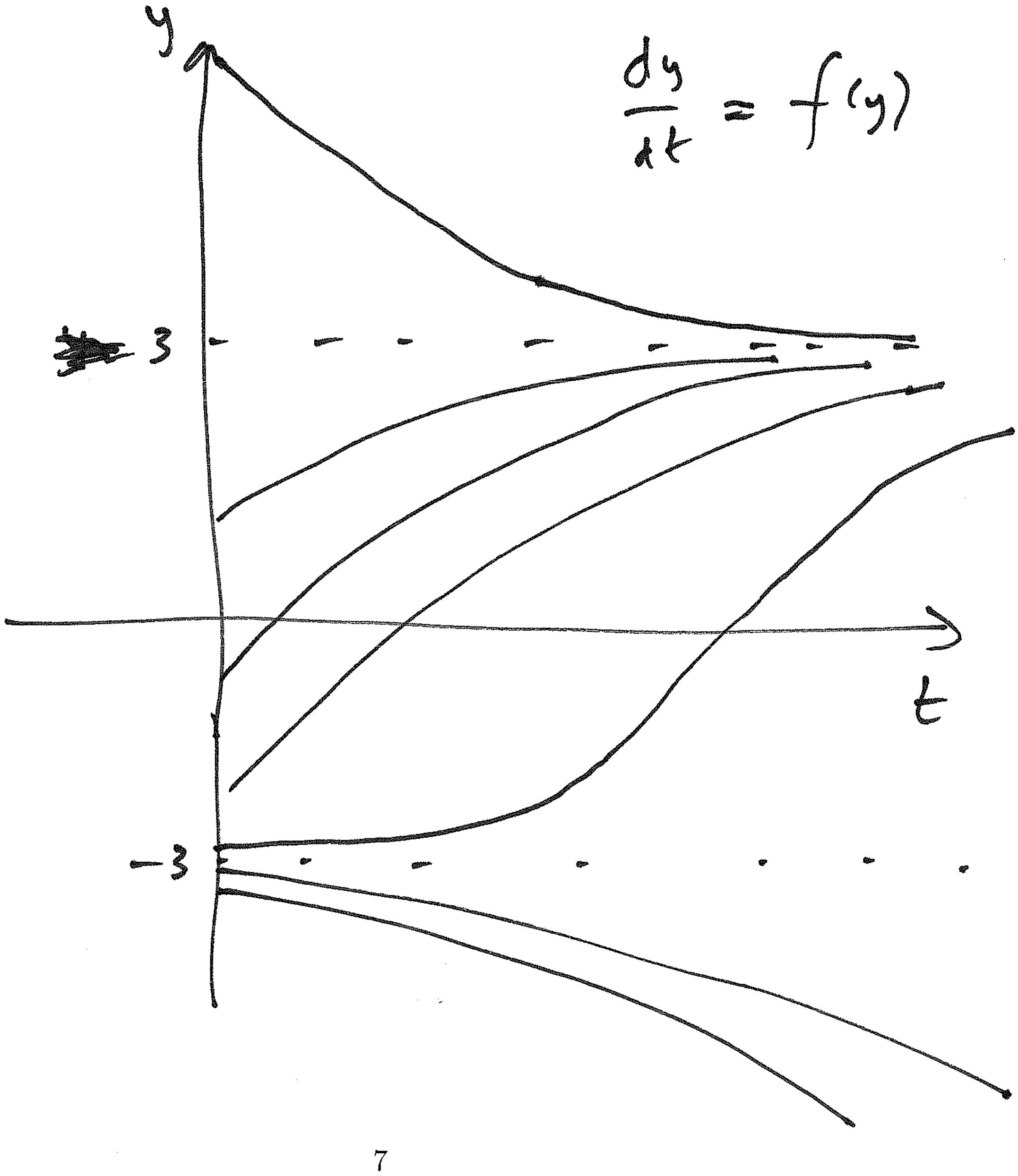


It is possible to sketch solutions to a DE just from the phase line.

Take previous example



## Longterm behaviour of solutions

In cases where the Uniqueness Theorem applies, a solution that tends to an equilibrium point does not reach the equilibrium point in finite time. We write

$$y(t) \longrightarrow y_0 \;\text{ as }\; t \longrightarrow \infty \text{ (or as } t \longrightarrow -\infty).$$

In contrast, a solution that tends to $+\infty$ or $-\infty$ may reach $\pm\infty$ in finite time or may never reach $\pm\infty$. We cannot tell which case we have from the phase line alone.

Example:

$$\frac{dy}{dt} = 1 \Rightarrow y = t + c$$

(takes forever to get to $\infty$)

Example:

$$\frac{dy}{dt} = 1 + y^2 , \quad y = \tan(t+c)$$

( gets to $\infty$ in finite time)

Comparing $\frac{dy}{dt} = 1$ & $\frac{dy}{dt} = y^2+1$

we get same phase line
but very diffrent behaviour

as $t$ ~~$\to \infty$~~ increases

~~is~~

These examples show that we cannot write

$$y(t) \to \pm\infty \quad \text{as} \quad t \to \infty \text{ (or as } t \to -\infty)$$

based on evidence from the phase line alone - we would need more information about the actual solutions before making such a statement.

Instead, based on phase lines, we make statements like

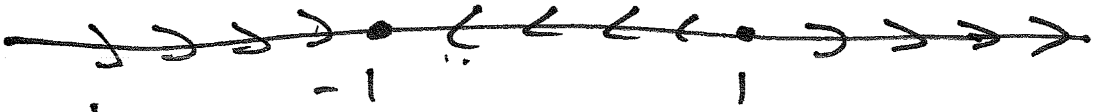
$$y(t) \to \infty \quad \text{as} \quad t \text{ increases}$$

or

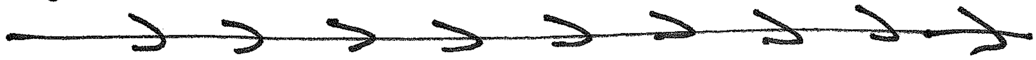
$$y(t) \to \infty \quad \text{as} \quad t \text{ decreases.}$$

Draw phase lines for

$$\frac{dy}{dt} = y^2 - 1$$

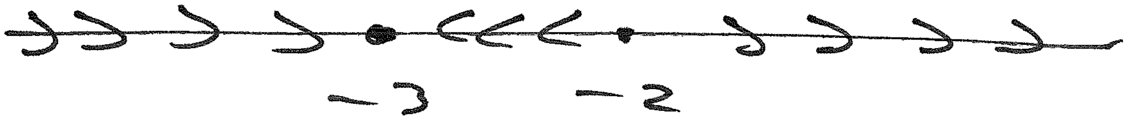



$$\frac{dy}{dt} = y^2 + 1$$



$$\frac{dy}{\ }$$

$$\frac{dy}{dt} = y^2 + 5y + 6$$



Main idea for today

For an autonomous differential equation

$$\frac{dy}{dt} = f(y)$$

it can be useful to sketch the phase line.

The phase line contains information about equilibrium solutions and whether other solutions are increasing or decreasing, but information about the speed with which solutions are changing is lost.

# Maths 260 Lecture 9

## Topics for today

Classification of equilibria

Linearization

## Reading for this lecture

BDH Section 1.6, pp 86-91

## Suggested Exercises

BDH Section 1.6: 1, 3, 5, 7, 13, 15, 17

## Reading for next lecture

BDH Section 1.7

## Today's handouts

Lecture 9 notes

Tutorial 3 question sheet

Assignment 2

## Classifying Equilibria

To draw the phase line for a DE

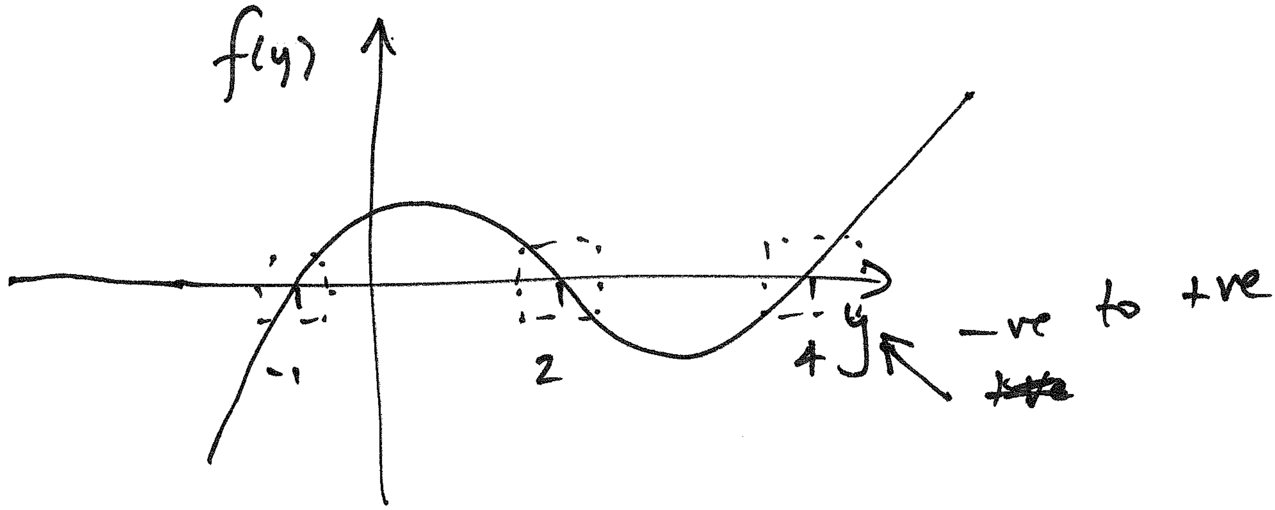
$$\frac{dy}{dt} = f(y)$$

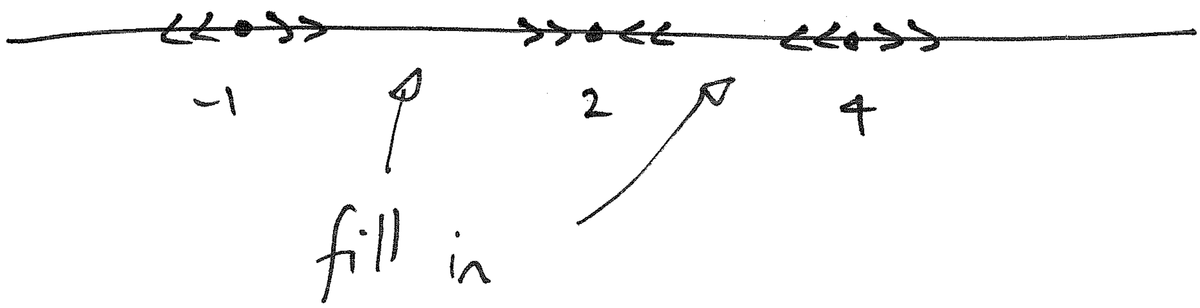
we need to know the positions of all equilibria, the intervals of $y$ where $f(y) > 0$ and the intervals of $y$ where $f(y) < 0$.

If $f$ is continuous, the sign of $f$ can only change at $y$ values where $f(y) = 0$, i.e., at equilibria.

Thus, the positions of the equilibria and the behaviour of solutions near each equilibrium is all we need to draw the phase line.

Example $\frac{dy}{dt} = f(y)$

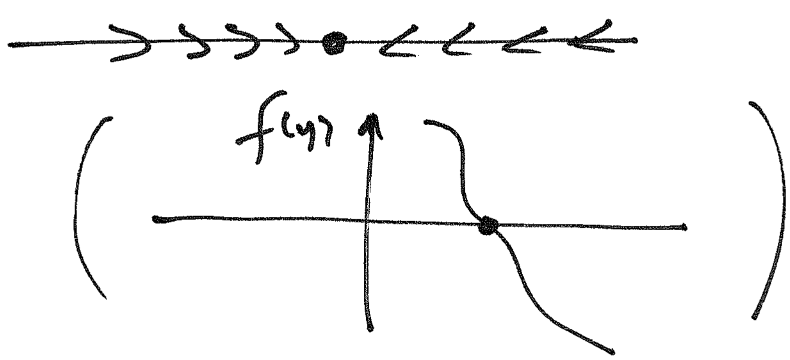


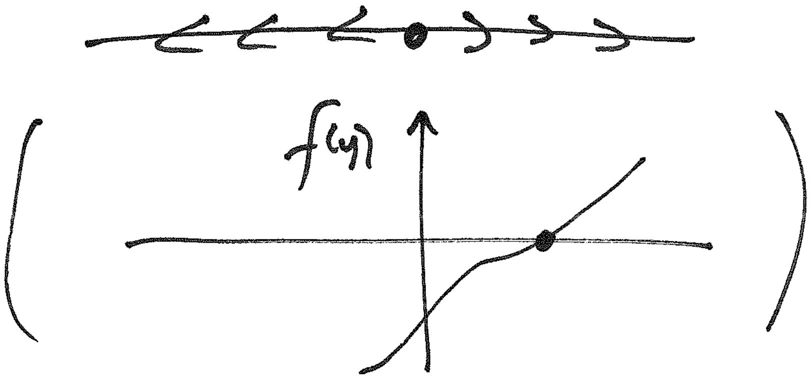
$\frac{dy}{dt} = f(y)$. Equilibrium point $f(y) = 0$ $\Rightarrow \frac{dy}{dt} = 0$

We classify equilibria according to the behaviour of nearby solutions.

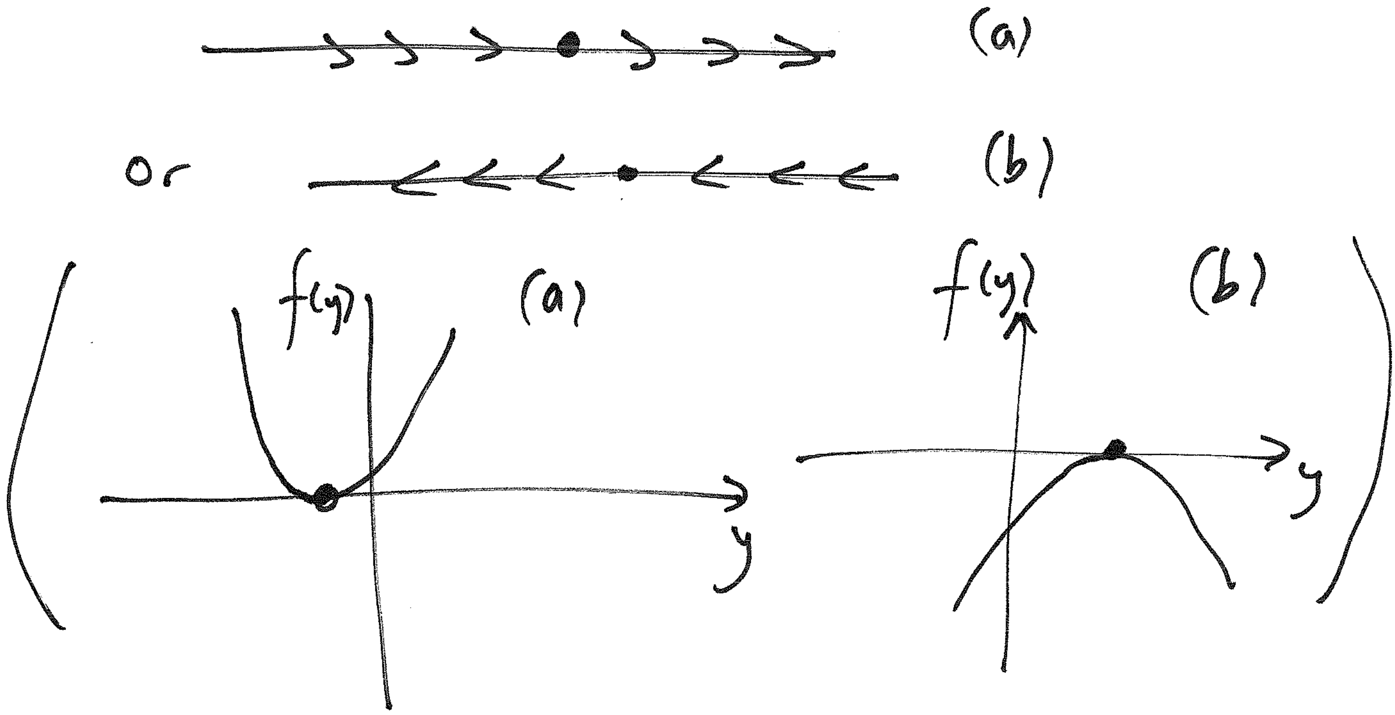
1. An equilibrium $y = a$ is a <u>sink</u> if any solution with initial condition sufficiently close to $a$ tends to $a$ as $t$ increases.




2. An equilibrium $y = b$ is a <u>source</u> if any solution with initial condition sufficiently close to $b$ tends away from $b$ as $t$ increases (which means nearby solutions diverge from $b$ as $t$ increases.)

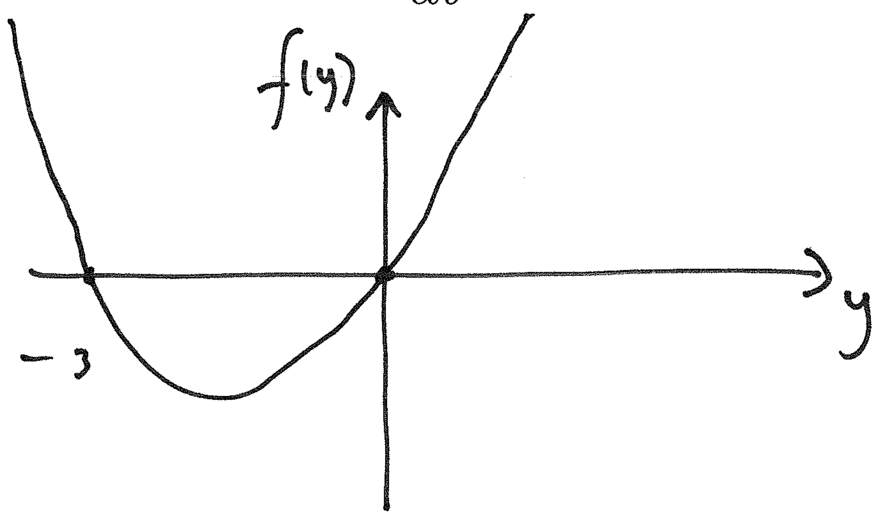


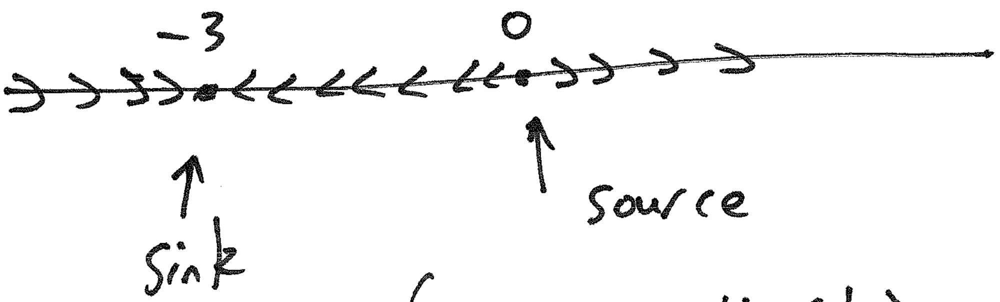
3. An equilibrium that is neither a sink nor a source is called a <u>node</u>.



Example 1.

$$\frac{dy}{dt} = y(3+y) \quad = f(y)$$




$(\ y_0 = y(t_0) = \text{initial } y\ )$

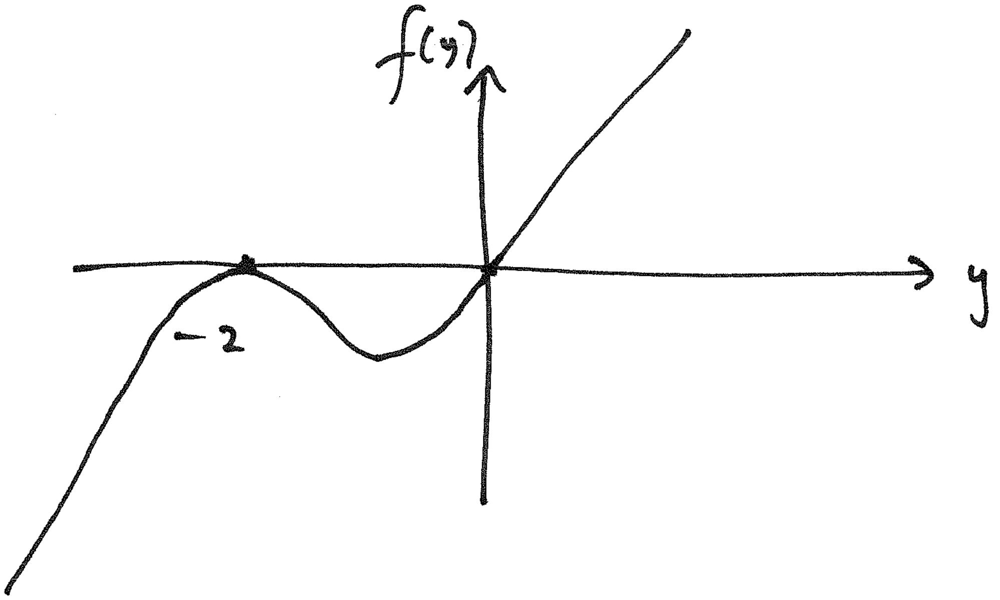
If $y_0 > 0$ $\quad y(t) \to \infty$ as $t$ increases

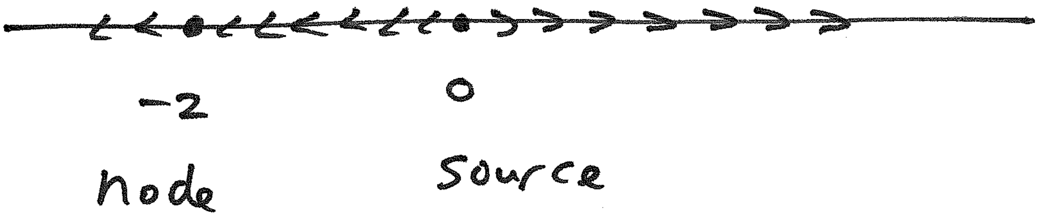
If $y_0 < 0$ $\quad y(t) \to -3$ at $t \to \infty$

If $y_0 = 0$ $\quad y(t) = 0$ for all $t$.

Example 2.

$$\frac{dy}{dt} = y(y+2)^2$$




If $y_0 > 0$ $y(t) \to \infty$ as $t$ increases

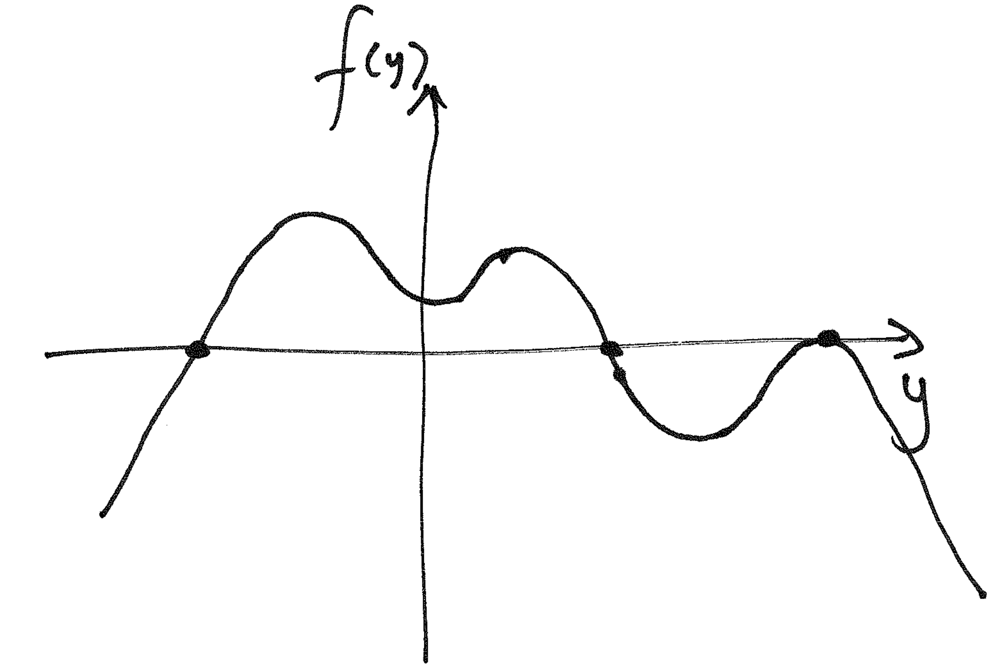
If $-2 < y_0 < 0$ $y(t) \to -2$ as $t \to \infty$

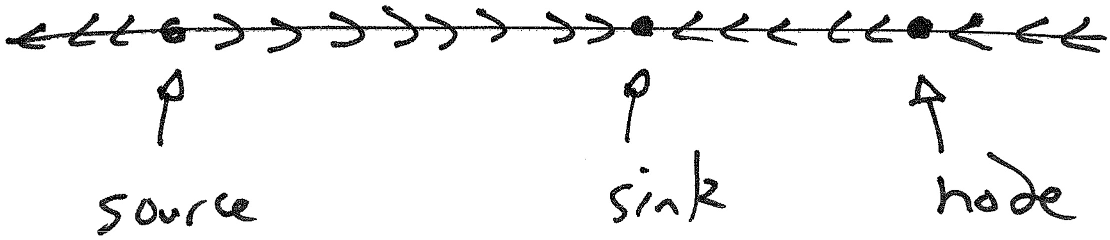
If $y_0 < -2$ $y(t) \to \infty$ as $t$ increases

Example 3.

$$\frac{dy}{dt} = f(y)$$

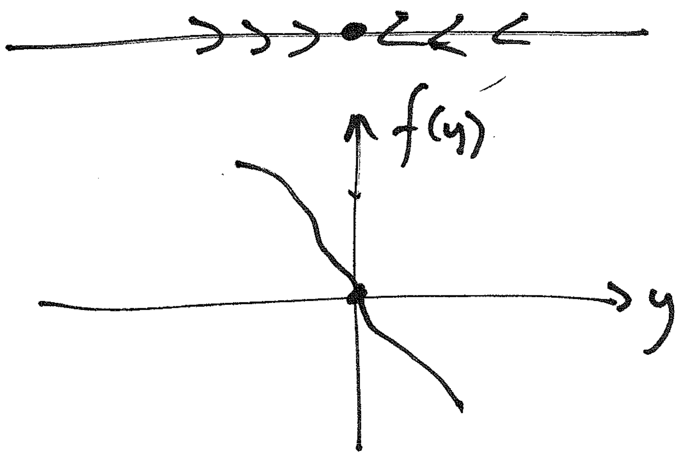
where $f(y)$ has graph shown.



Linearization

If $y_0$ is an equilibrium solution of $\frac{dy}{dt} = f(y)$ and is a sink, then the phase line near $y_0$ looks like

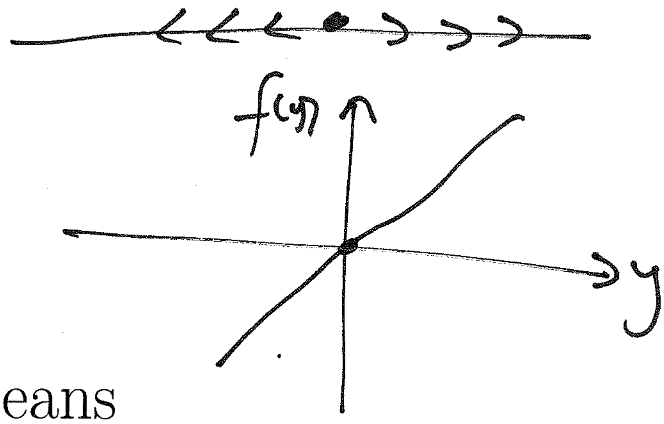



which means

- $f(y) > 0$ if $y < y_0$
- $f(y) < 0$ if $y > y_0$
- $f(y_0) = 0$

So $f(y)$ is a decreasing function near $y_0$.

If $y_0$ is an equilibrium solution of $\frac{dy}{dt} = f(y)$ and is a <u>source</u>, then the phase line near $y_0$ looks like




which means

- $f(y) < 0$ if $y < y_0$
- $f(y) > 0$ if $y > y_0$
- $f(y_0) = 0$

So $f(y)$ is a increasing function near $y_0$.

These examples motivate the following theorem:

**Linearization Theorem**

Suppose that $y = y_0$ is an equilibrium point of the DE

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ and $\partial f/\partial y$ are both continuous.

1. If $f'(y_0) < 0$, then $y_0$ is a sink.
2. If $f'(y_0) > 0$, then $y_0$ is a source.
3. If $f'(y_0) = 0$, or if $f'(y_0)$ does not exist, then we need additional information to determine the type of $y_0$.

**Note:** In the last case, the equilibrium may be a node or a sink or a source.

Example 1:

For the DE

$$\frac{dy}{dt} = y^2(y-2)(y+2)$$

find all equilibrium solutions and classify them using the linearization theorem.

Equilibrium points $y = 0, -2, 2$

$$f(y) = y^2(y-2)(y+2)$$

$$= y^2(y^2-4) = y^4 - 4y^2$$

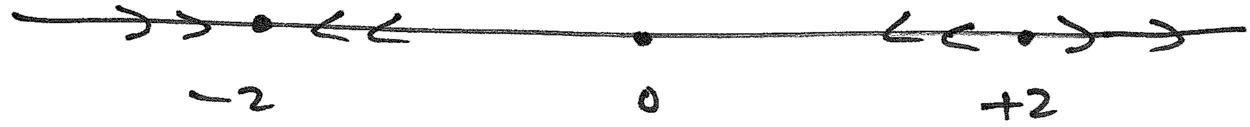
$$\frac{df}{dy} = 4y^3 - 8y$$

$$\left.\frac{df}{dy}\right|_{y=-2} = -16 \Rightarrow \text{eq. pt. sink}$$

$$\left.\frac{df}{dy}\right|_{y=0} = 0 \Rightarrow$$ we can not tell ~~eq pt source~~

$$\left.\frac{df}{dy}\right|_{y=2} = 16 \Rightarrow \text{eq. pt. source}$$

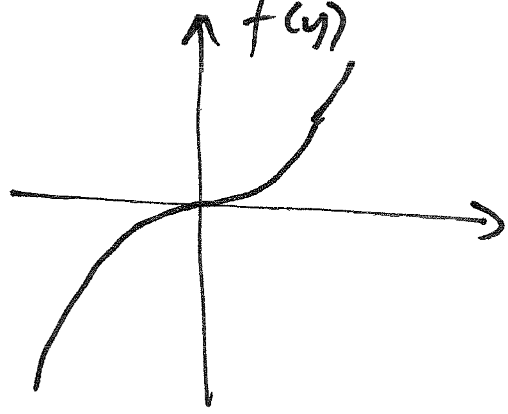
Phase line

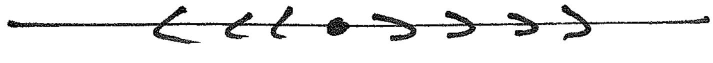



( we can tell from picture that $y=0$ is a node but it does not follow from $f'(y)=0$ )

e.g.

$$\frac{dy}{dt} = y^3$$



$y=0$ is a source but $f'(y)\big|_{y=0} = 0$

Example 2:

Consider the following population model:

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{200}\right)\left(\frac{P}{50} - 1\right) = f(P)$$

Classify the equilibria, draw the phase line and sketch some solutions for $P$.

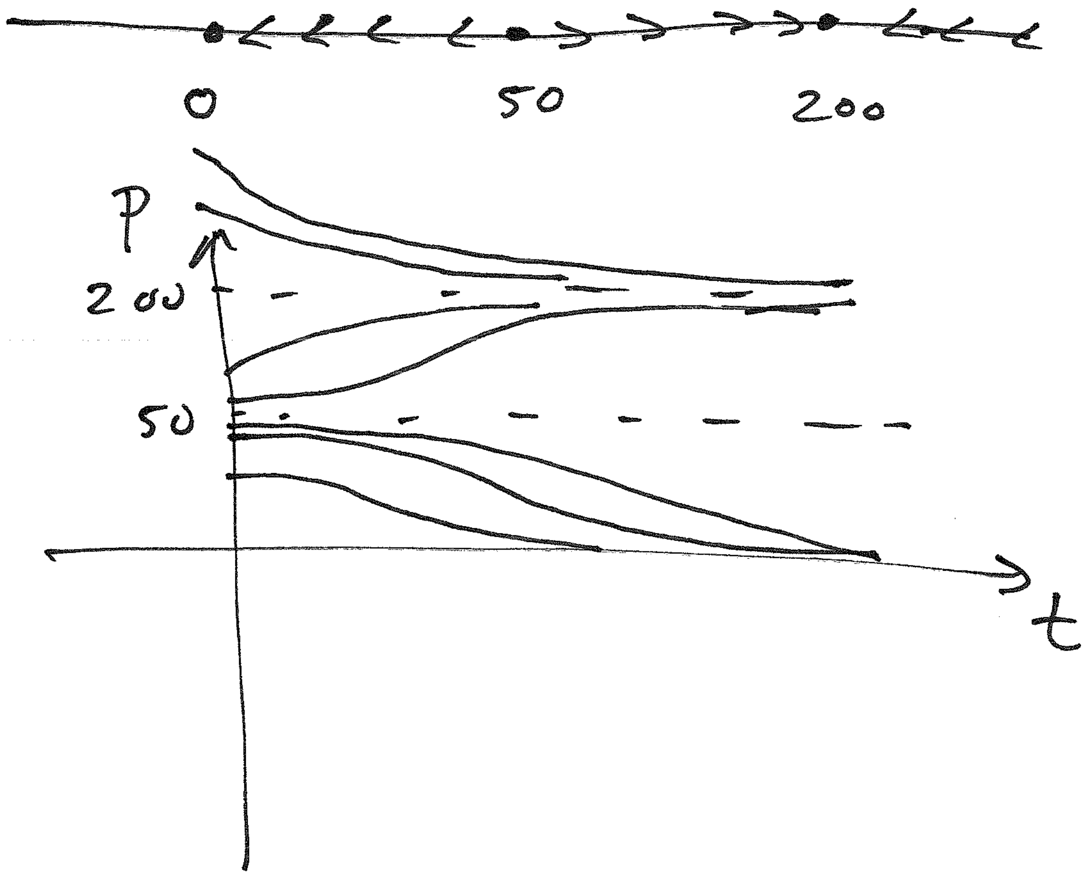
eq. pt. $P = 0, 200, 50$

$$\frac{df}{dP} = 0.3\left(1 - P/200\right)\left(P/50 - 1\right) + \frac{0.3P}{-200}\left(P/50 - 1\right) + \frac{0.3P}{50}\left(1 - P/200\right)$$

$$\left.\frac{dP}{dt}\right|_{P=0} = -0.3 \Rightarrow \text{sink}$$

$$\left.\frac{dP}{dt}\right|_{P=200} = -0.9 \Rightarrow \text{sink}$$

$$\left.\frac{dP}{dt}\right|_{P=50} = 0.225 \Rightarrow \text{source}$$




ie. $P_0 < 50$ $P(t) \to 0$

and $P_0 > 50$ $P(t) \to 200$.

Important ideas from today

Equilibria are classified as sink, source or node depending on the behaviour of nearby solutions.

Linearization - we can sometimes use $df/dy$ to classify equilibria.