Maths 260 Lecture 27

Topic for today

Sketching phase portraits for nonlinear systems

Reading for this lecture

BDH Section 5.2

Suggested exercises

Tutorial 8 questions

Reading for next lecture

BDH Section 5.5

Today's handout

Lecture 26 notes

Sketching phase portraits for nonlinear systems

We have learnt two methods for obtaining information about solutions to nonlinear systems:

1. Linearisation can give information about the behaviour of solutions near an equilibrium solution.
2. The method of nullclines gives information about where in the phase plane solution curves are horizontal and vertical. From this we can deduce where in the phase plane solutions move up, down, left or right.

We use both of these methods to sketch phase portraits for nonlinear systems.

Outline of method

To sketch a phase portrait, it can be helpful to follow some or all of the following steps.

1. Find all equilibria. Where possible, use linearisation to determine their types (e.g., saddle, spiral source).
2. Draw the nullclines. Determine the direction of the vector field in the regions between nullclines. Determine the direction of the vector field on the nullclines.
3. Sketch some representative solution curves. Make sure the solution curves you sketch go in the directions determined by the nullclines and behave like the appropriate linearised system near any equilibrium.

Note: Nullclines are **not** usually solution curves.

Example 1

Sketch the phase portrait for the system

$$\frac{dx}{dt} = 2 - x - y \quad = f(x,y)$$
$$\frac{dy}{dt} = x^2 - y \quad = g(x,y)$$

Determine the long term behaviour of the solutions through $(x, y) = (1, 2)$ and $(-3, 4)$.

Equilibria $f = 0$ & $g = 0$

i.e. $2 - x - y = 0 \quad (*)$

$x^2 - y = 0$

Substitute $y = x^2$ into

$\Rightarrow \quad 2 - x - x^2 = 0 \quad \Rightarrow x = 1, -2$

$\Rightarrow$ eq. points $(1, 1)$, $(-2, 4)$

Jacobian $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2x & -1 \end{pmatrix}$

Solutions near $(1, 1)$

$$J_{(1,1)} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$$

Find eigenvalues & eigenvectors

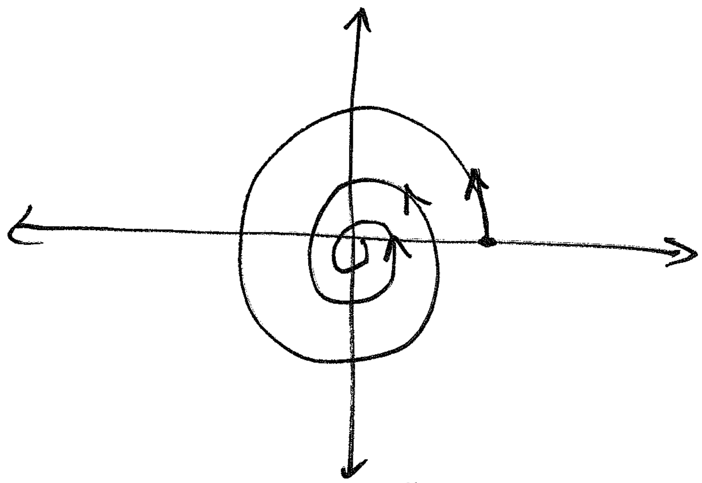
$$\lambda = \underbrace{-1 \pm i\sqrt{8}}_{\cancel{2}} /2$$

$$= -1 + i\sqrt{2}$$

We do not need eigenvectors.

Eq. point at $(1, 1)$ is spiral sink

At $(1,0)$ $\quad \frac{dr}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$



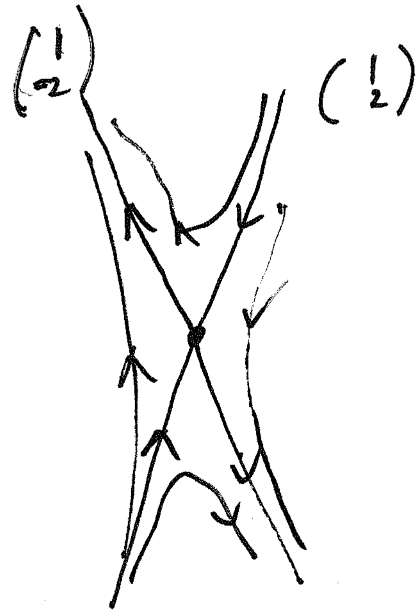
Solutions near $(-2, 4)$

$$J(-2,4) = \begin{pmatrix} -1 & -1 \\ -4 & -1 \end{pmatrix}$$

Find eigenvalues & eigenvectors.

$1 \leftrightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\qquad -3 \leftrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Saddle

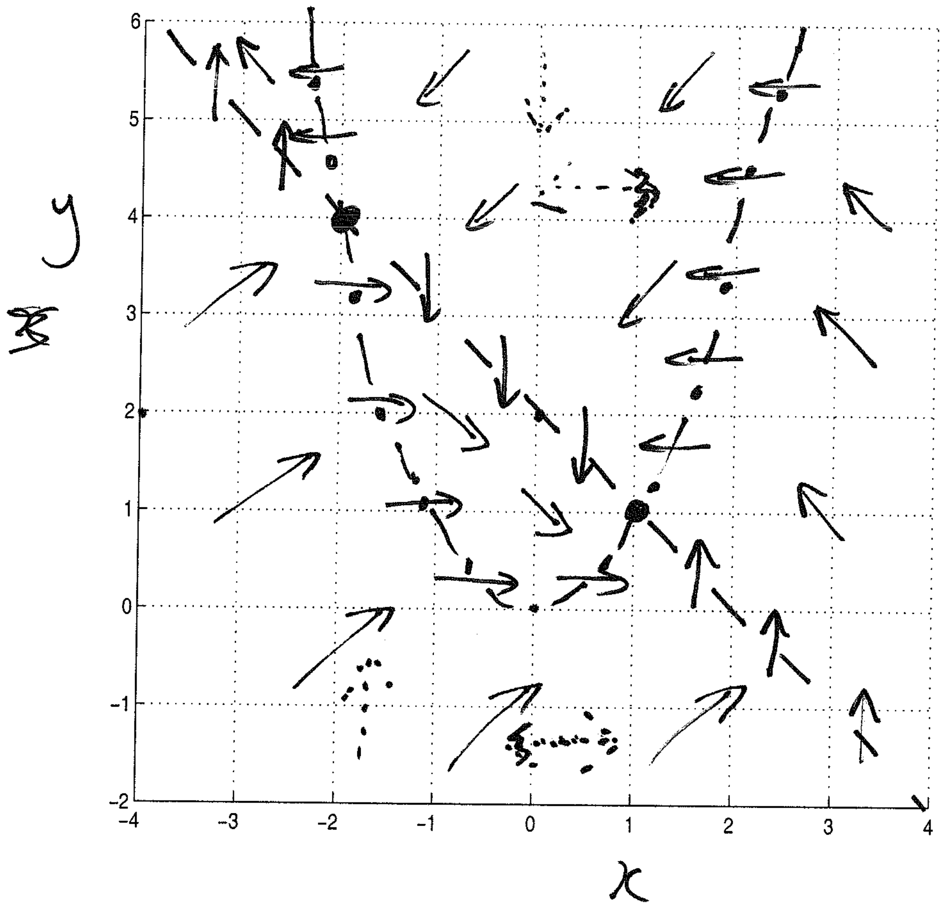


Nullclines

$$\frac{dx}{dt} = 2 - x - y$$

$$\frac{dy}{dt} = x^2 - y$$

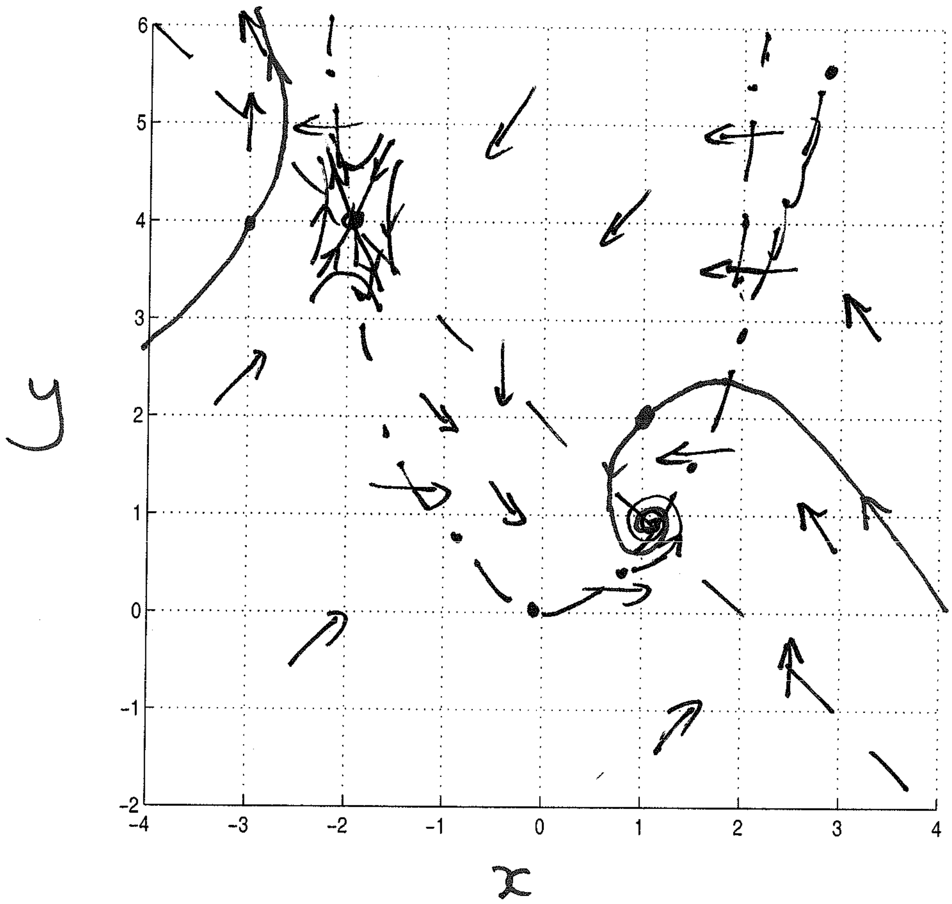
Directions of solutions on and between nullclines



— — — x - nullcline $\quad 2 - x - y = 0$ or $y = 2 - x$

— • — y - nullcline $\quad x^2 - y = 0$ or $y = x^2$

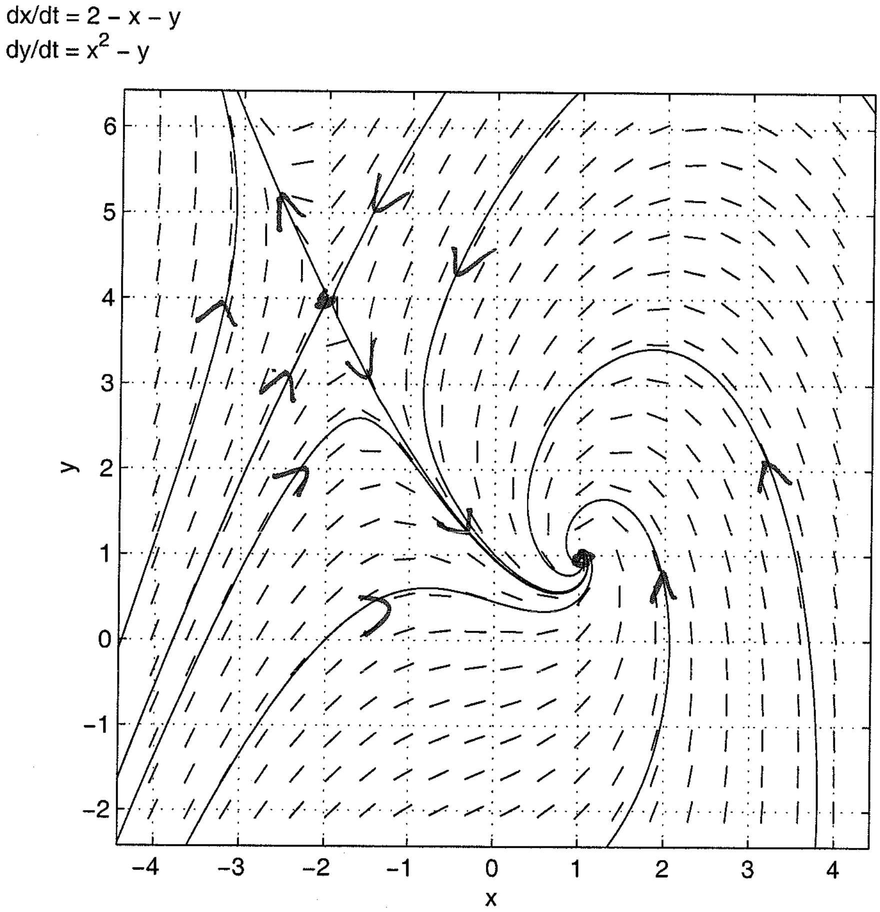
Sketch of phase portrait



Behaviour of solution through $(1, 2)$

Behaviour of solution through $(-3, 4)$

Phase portrait from *pplane*



Example 2

Sketch the phase portrait for the system

$$\frac{dx}{dt} = x(x-1) \quad = f$$
$$\frac{dy}{dt} = x^2 - y \quad = g$$

Determine the long term behaviour of the solution through $(x, y) = (-1, 0)$, $(0.8, 0)$ and $(1, 3)$.

Equilibria $\quad f = x(x-1) = 0$

$$g = x^2 - y = 0$$

eq. pts $\quad (0,0), \ (1,1)$

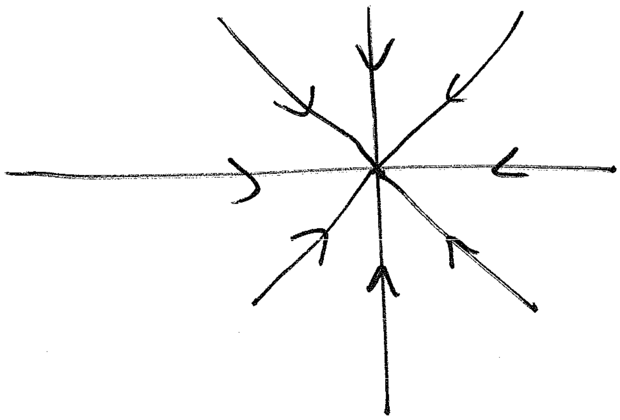
Jacobian

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x-1 & 0 \\ 2x & -1 \end{pmatrix}$$

Solutions near $(0, 0)$

$$J_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \lambda\ -1 \Leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad -1 \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

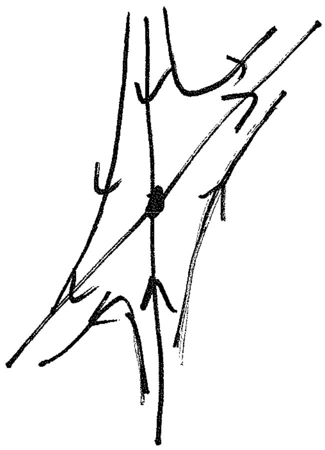
sink



Solutions near $(1, 1)$

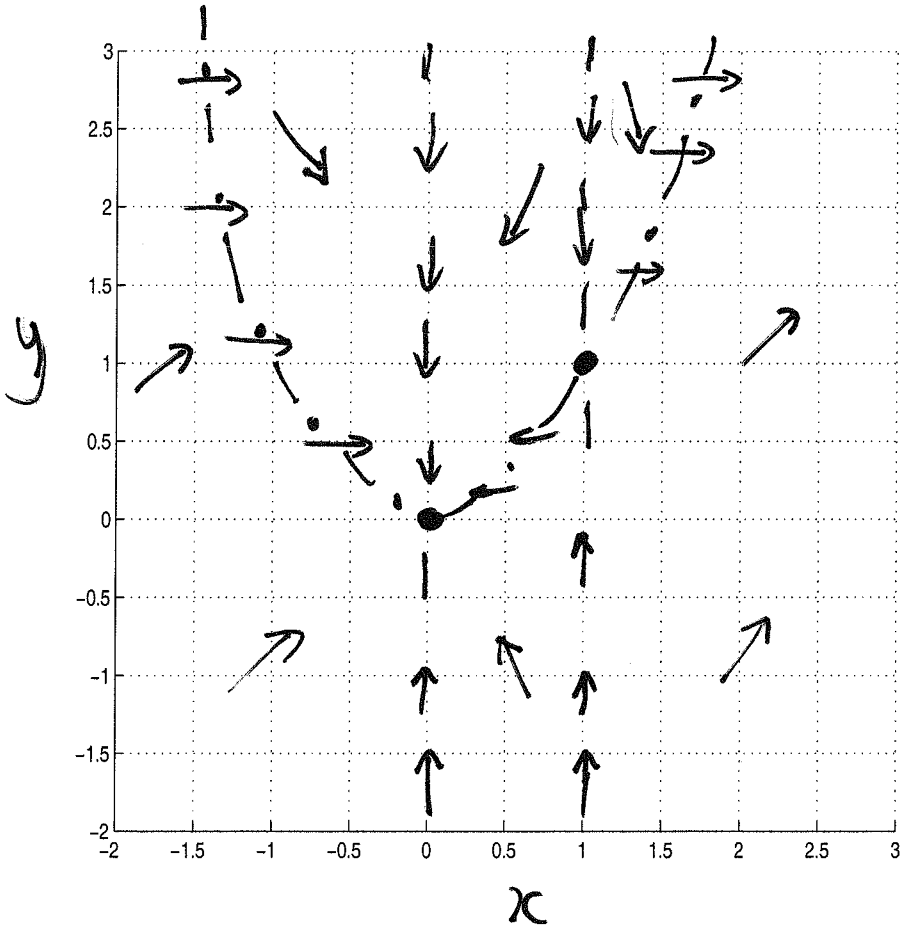
$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \qquad 1 \Leftrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad -1 \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

saddle



Nullclines

Directions of solutions on and between nullclines



— — — x- nullcline $\quad x\,(x-1)=0$

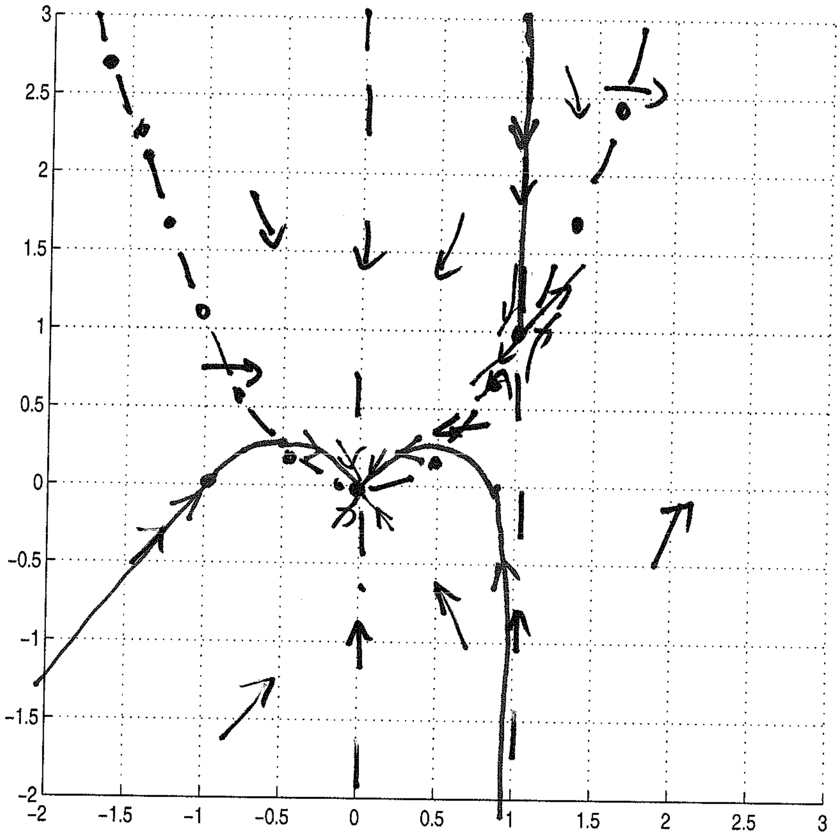
$\Rightarrow x=0 \;\&\; x=1$

— • — • y - nullcline $\Rightarrow x^2-y=0$

$y=x^2$

Sketch of phase portrait

— — — x - null
— • — y - null

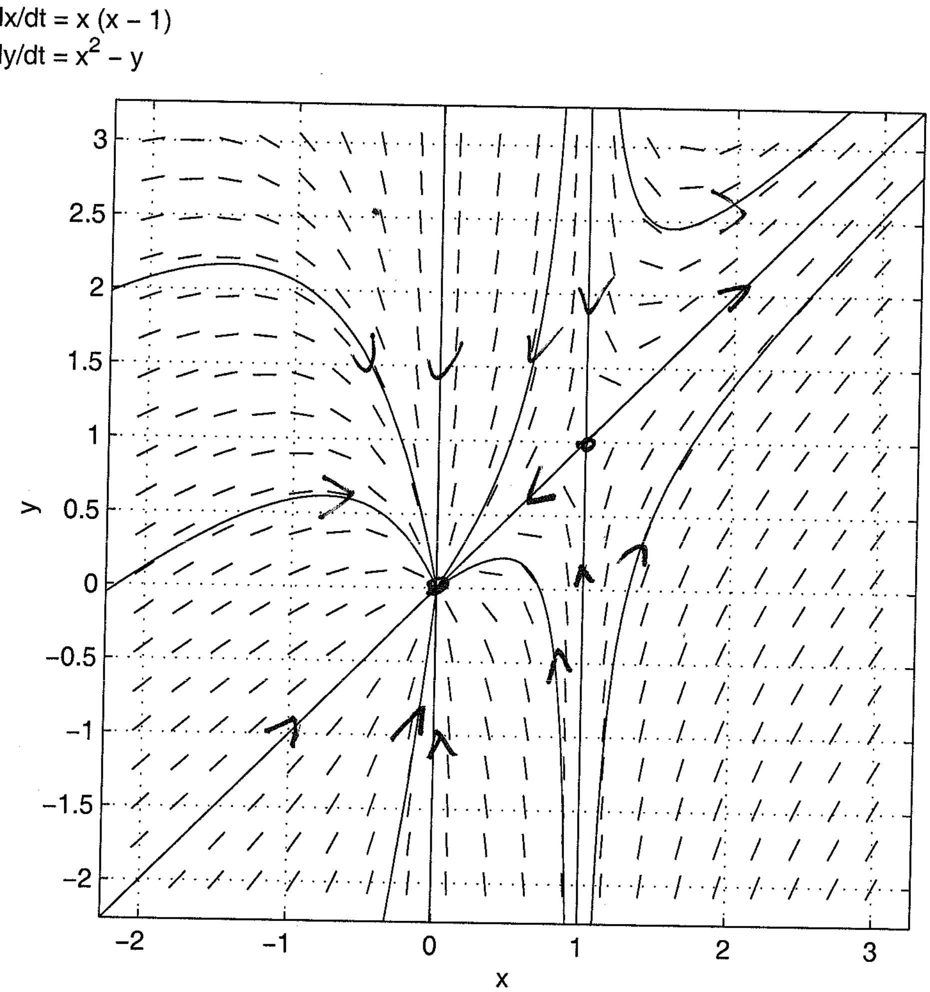


Behaviour of solution through $(-1, 0)$

Behaviour of solution through $(0.8, 0)$

Behaviour of solution through $(1, 3)$

Phase portrait from *pplane*



Maths 260 Lecture 28

Topic for today
Periodic solutions

Reading for this lecture
BDH Section 5.5

Suggested exercises

Reading for next lecture
BDH Section 3.6

Today's handouts
Lecture 27 notes

Example

Sketch the phase portrait for the system

$$\frac{dx}{dt} = -y \qquad = f$$

$$\frac{dy}{dt} = 1 - 0.9y - x^2 - xy \quad = g$$

Equilibria

$(1, 0)$ & $(-1, 0)$

Jacobian

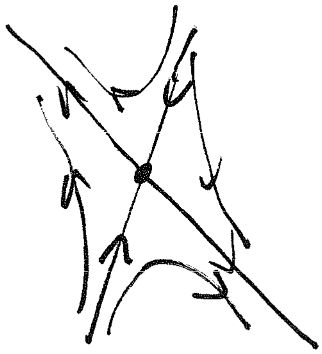
$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -2x-y & -0.9-x \end{pmatrix}$$

Solutions near $(1, 0)$

$$J_{(1,0)} = \begin{pmatrix} 0 & -1 \\ -2 & -1.9 \end{pmatrix}$$

$$-2.6 \Leftrightarrow \begin{pmatrix} 0.4 \\ 0.9 \end{pmatrix} \qquad 0.7 \Leftrightarrow \begin{pmatrix} 0.8 \\ -0.6 \end{pmatrix}$$

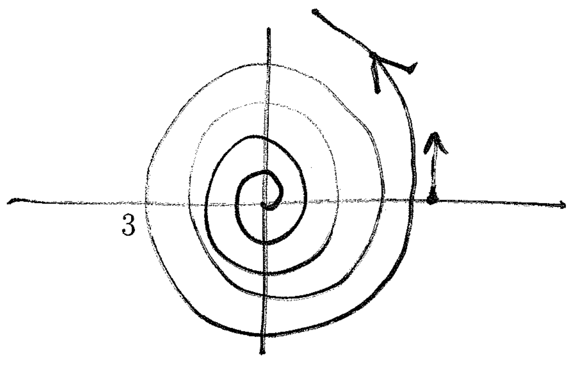


Solutions near $(-1, 0)$

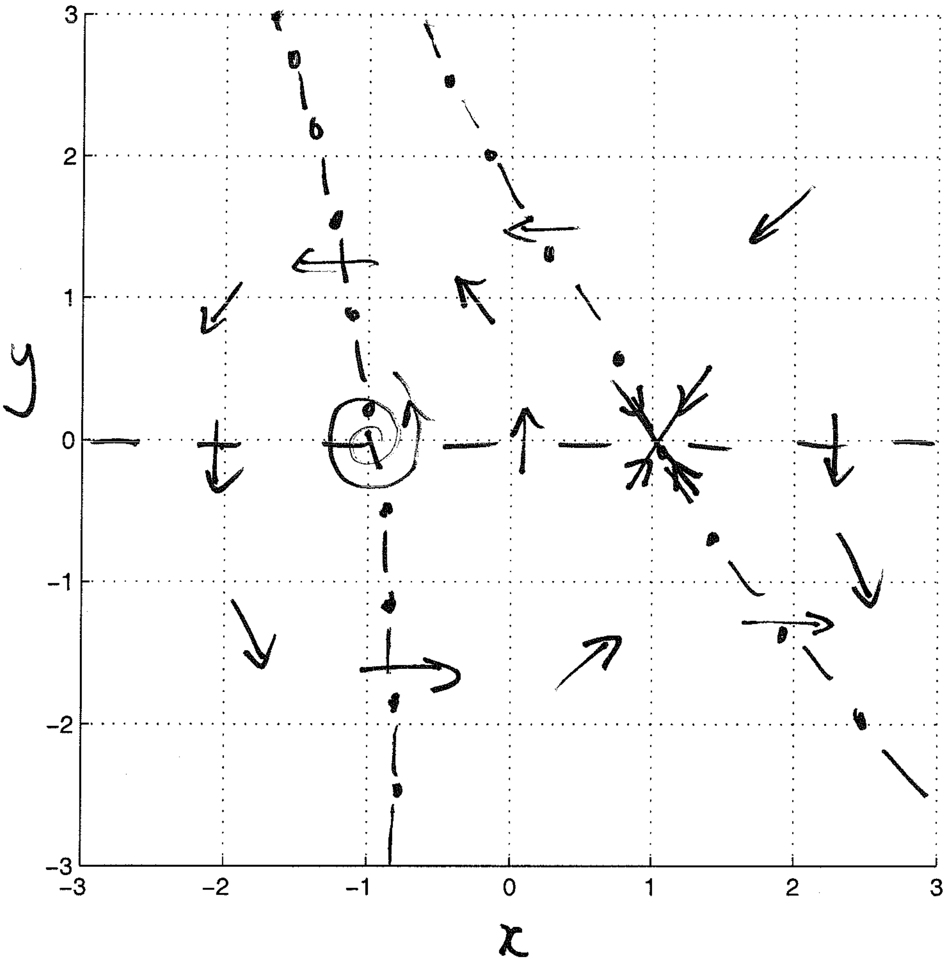
$$J_{(-1,0)} = \begin{pmatrix} 0 & -1 \\ 2 & 0.1 \end{pmatrix}$$

$$\lambda = 0.05 \pm \cancel{0}1.4i$$

$$\left.\frac{dr}{dt}\right|_{(1,0)} = \begin{pmatrix} 0 & -1 \\ 2 & 0.1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



Nullclines



$--$ x nullcline $-y=0$

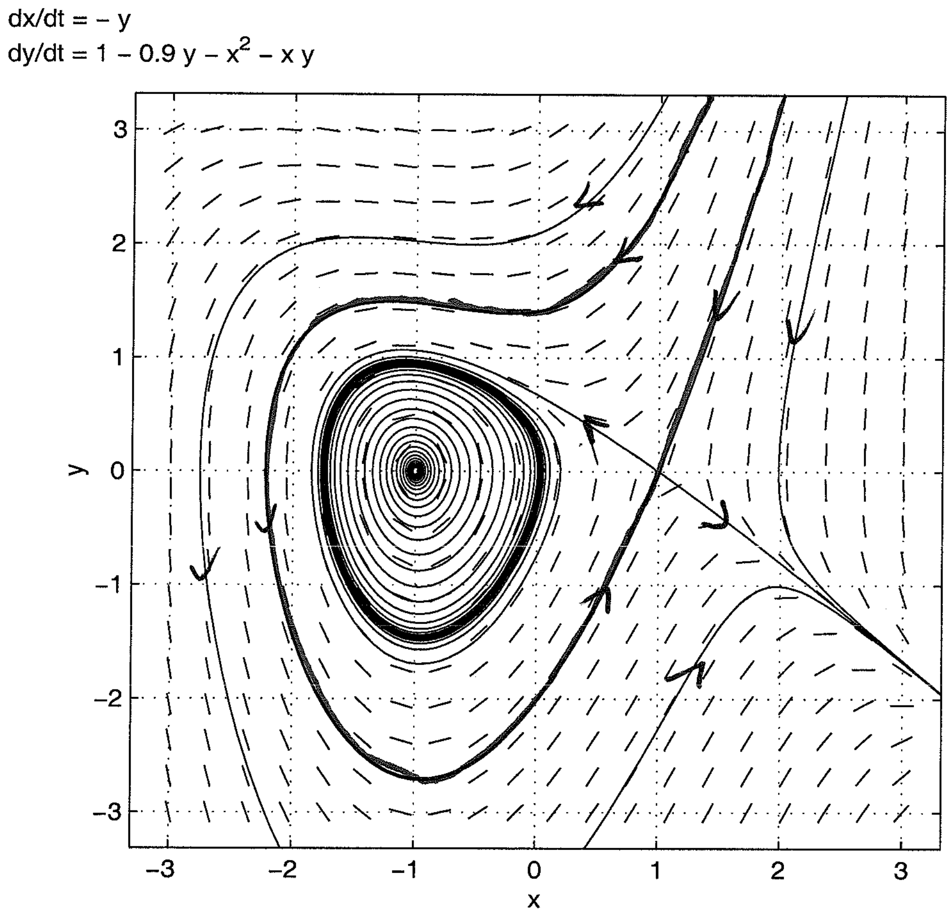
$-\cdot-\cdot$ y nullcline

$$1-0.9y-x^2-xy=0$$

$$\Rightarrow \quad 1-x^2 = 0.9y+xy$$

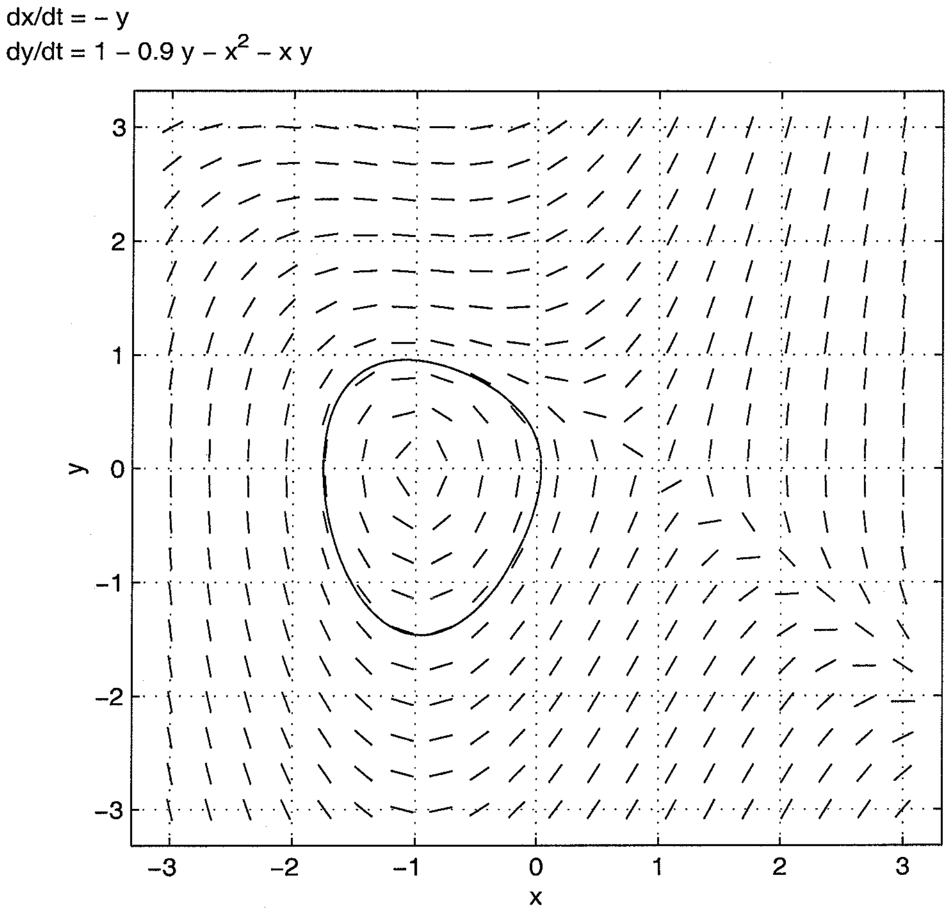
$$y = \frac{1-x^2}{0.9+x}$$

Phase portrait from *pplane*



Note how close together solution curves get in a band around the equilibrium at $(-1, 0)$.

Careful use of *pplane* gives the following solution curve:



We see there is a closed solution curve passing close to the origin. This curve corresponds to a *periodic solution* of the system, i.e., a solution for which each dependent variable is a periodic function of time.

Example : Use *pplane* to investigate the qualitative changes in the behaviour of solutions to

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = \lambda - 0.9y - x^2 - xy$$

that occur as λ is varied in the interval $[-3, 3]$.

(just studied $\lambda = 1$)

Some advanced features of *pplane* are useful for investigating this system. In particular, *pplane* can be used to do the following.

1. Plot nullclines. Select the 'Nullclines' option in the lower right of the 'Setup' window.

2. Find equilibria and determine their type. Select the option 'Find an equilibrium' from the 'Solutions' menu on the 'Display' window, move the cursor to a place in the display window near where you expect the equilibrium to be and click.

3. Plot solutions for t increasing only. In the 'Display' window, pull down the 'Options' menu, pick 'Solution direction' and then select 'Forward'.