Maths 260 Lecture 25

Topic for today

Non-linear systems: linearisation near equilibria

Reading for this lecture

BDH Section 5.1

Suggested exercises

BDH Section 5.1; 1, 3, 7, 9, 11

Reading for next lecture

BDH Section 5.2

Today's handout

Lecture 24 notes

2.10 Nonlinear Systems

Consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^3 - \frac{1}{2}y \end{aligned}\right\}$$

t does not appear in R.H.S.

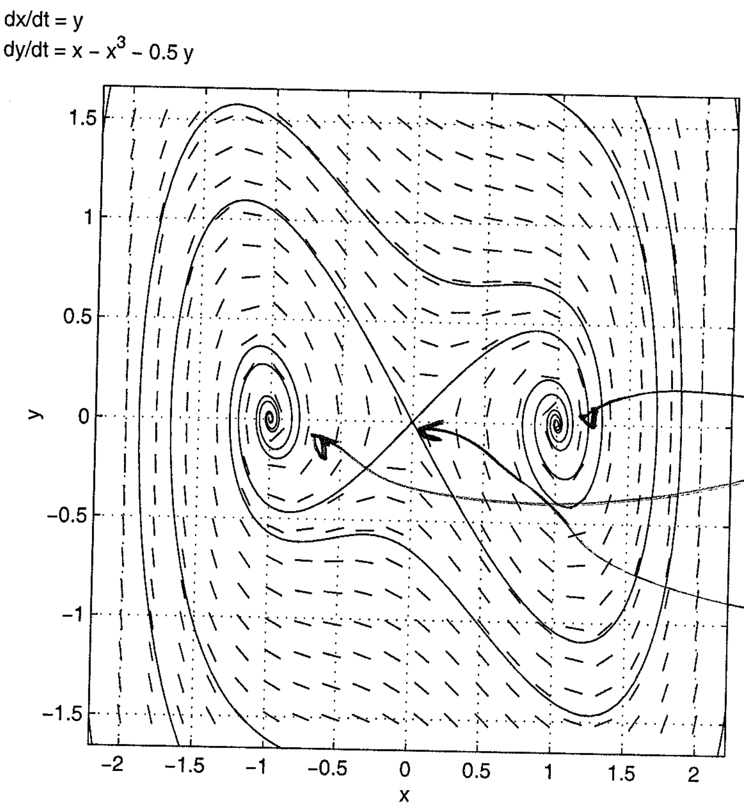
Equilibrium solutions:

$\frac{dx}{dt} = 0$ & $\frac{dy}{dt} = 0$

$$\Rightarrow \left.\begin{aligned} &y = 0 \\ &x - x^3 - \tfrac{1}{2}y = 0 \end{aligned}\right\} \Rightarrow (0,0)\ (-1,0)\ (1,0)$$

are equilibrium points

Slope field and some solutions



spiral sink

saddle

We can understand the saddle-like nature of $(0,0)$ if we approximate the nonlinear system by a linear system.

For x, y very close to zero, x^3 is much smaller than x or y.

So we can ignore x^3 term in the nonlinear system, and approximate the behaviour of the nonlinear system near $(0,0)$ with the linear system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - \frac{1}{2}y \quad - \cancel{x^3} \end{aligned}$$

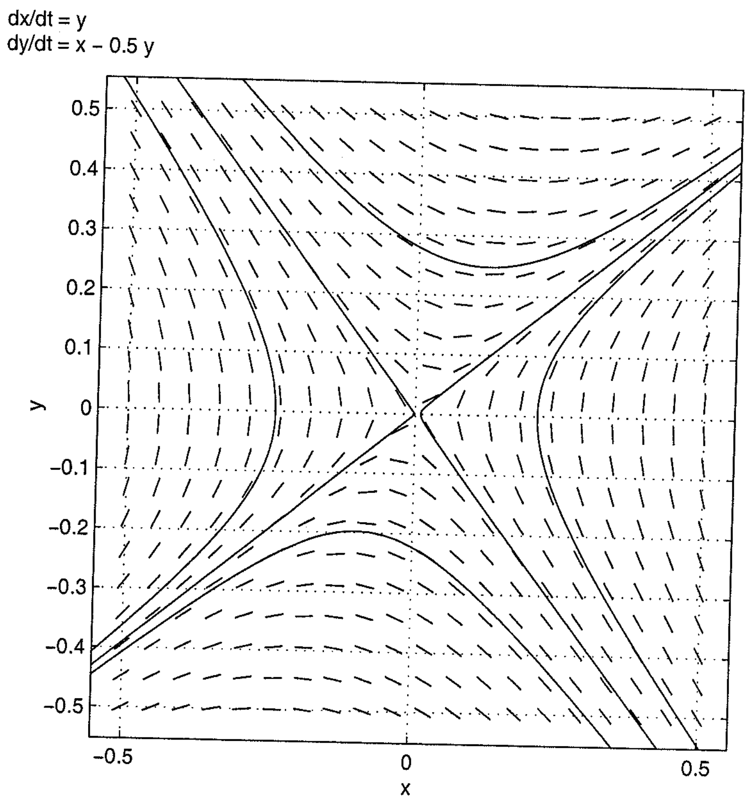
i.e.,

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

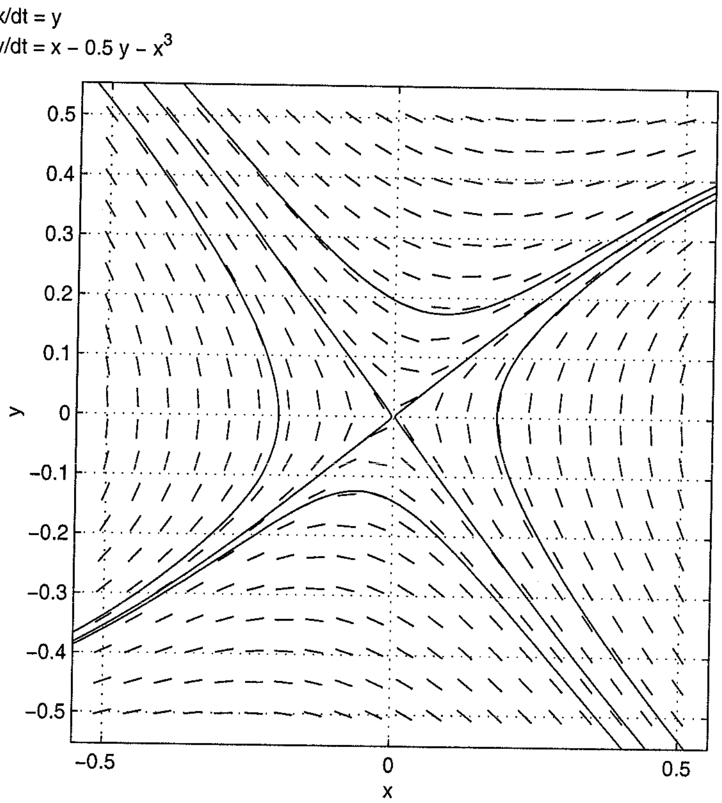
The eigenvalues of matrix A are 0.78 and -1.28 so the equilibrium at the origin of linear system is a saddle.

The following pictures show the slope field and solutions for the linear system and for the nonlinear system near the origin. Note that the linear system is a good approximation near the origin but is hopeless away from the origin.

Slope field and solutions for linear system



Slope field and solutions for nonlinear system



This procedure is called **linearisation**:

Near an equilibrium, approximate the nonlinear system by an appropriate linear system.

For initial conditions near the equilibrium, solutions of the nonlinear system stay close to solutions of the approximate linear system, at least for some interval of time.

Thus, the type of equilibrium at the origin in linearised system gives information about the type of the corresponding equilibrium in the nonlinear system.

Returning to original example, consider equilibria at $(1, 0)$ and $(-1, 0)$.

To approximate behaviour near $(1, 0)$ by a linear system, we need to first shift the equilibrium to the origin – because linear systems usually only have an equilibrium at the origin.

Change the coordinates as follows:

$$u = x - 1$$
$$v = y$$

so the equilibrium $(x, y) = (1, 0)$ is now at $(u, v) = (0, 0)$.

Then the system becomes:

$$\frac{du}{dt} = \frac{dx}{dt} = \cancel{x - x^3 - \frac{1}{2}y = (u+1) - u+1}\ y = v$$

$$\frac{dv}{dt} = \frac{dy}{dt} = x - x^3 - \frac{1}{2}y = (u+1) - (u+1)^3 - \frac{1}{2}v$$
$$= -2u - 3u^2 - u^3 - \frac{1}{2}v$$

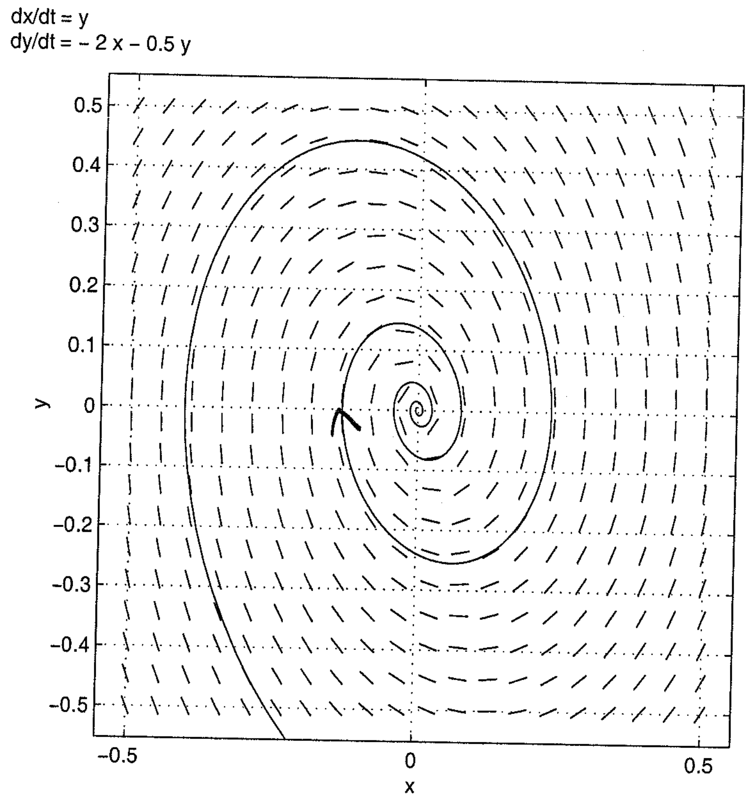
For u and v small, $-3u^2$ and $-u^3$ are very, very small. Ignore these nonlinear terms and approximate system by:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

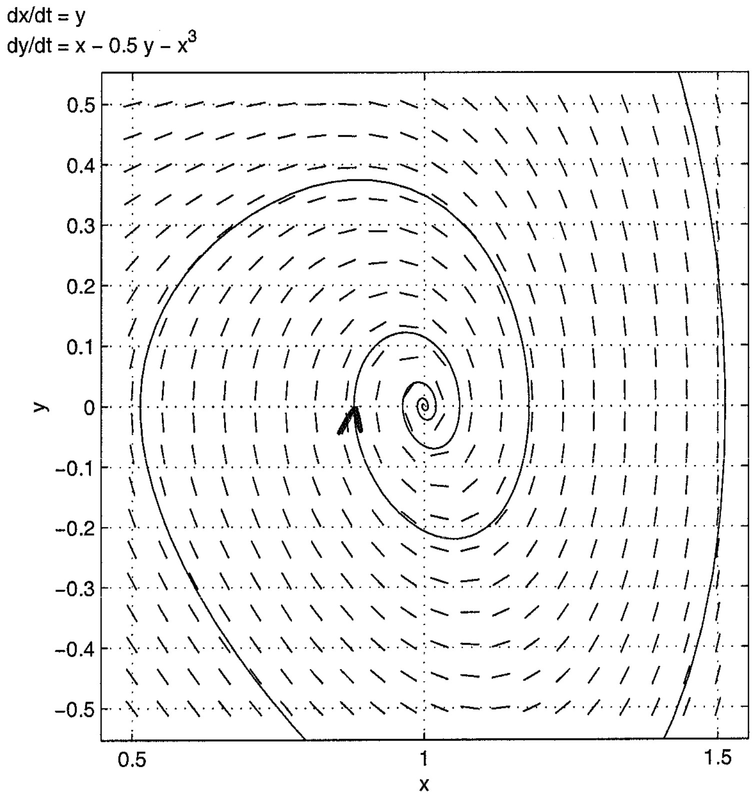
Eigenvalues are $-\frac{1}{4} \pm \frac{1}{4}\sqrt{31}i$. So origin is a spiral sink in the linear approximation.

The following pictures illustrate the similarity between the phase portrait near the equilibrium at $(1, 0)$ in the nonlinear system and the phase portrait for the linearised system.

Phase portrait for linearised system



Phase portrait near $(1, 0)$ in nonlinear system



Similar calculations give similar results for the equilibrium at $(-1, 0)$.

More generally, if the system

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

has an equilibrium at (x_0, y_0), we can construct a linear approximation to the system for x and y values near (x_0, y_0) as follows:

First move the equilibrium to the origin. Write $u = x - x_0, v = y - y_0$.

The nonlinear equations in the new coordinates are:

$$\frac{du}{dt} = \frac{dx}{dt} = f(x, y) = f(x_0 + u, y_0 + v)$$

$$\frac{dv}{dt} = \frac{dy}{dt} = g(x, y) = g(x_0 + u, y_0 + v)$$

Now we use Taylor expansion to rewrite f and g:

constant

$$f(x_0 + u, y_0 + v) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right] u$$

$$+ \left[\frac{\partial f}{\partial y}(x_0, y_0)\right] v + h.o.t$$

$$g(x_0 + u, y_0 + v) = g(x_0, y_0) + \left[\frac{\partial g}{\partial x}(x_0, y_0)\right] u$$

$$+ \left[\frac{\partial g}{\partial y}(x_0, y_0)\right] v + h.o.t$$

If we ignore the higher order terms and note that $f(x_0, y_0) = g(x_0, y_0) = 0$, then we get an approximate linear system:

$$\begin{pmatrix} \frac{du}{dt} \\ \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ \\ v \end{pmatrix} \qquad (1)$$

i.e., the behaviour of solutions to the nonlinear system near the equilibrium (x_0, y_0) can be approximated by the behaviour of solutions in the linearised system (1).

The matrix of partial derivatives in (1) is called the **Jacobian** matrix.

e.g.

$$\frac{dx}{dt} = y = f(x, y)$$

$$\frac{dy}{dt} = x - x^3 - \frac{1}{2}y = g(x, y)$$

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & -1/2 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -1/2 \end{pmatrix}$$

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ -2 & -1/2 \end{pmatrix}$$

$$J(-1,0) = \begin{pmatrix} 0 & 1 \\ -2 & -1/2 \end{pmatrix}$$

Example: Consider the system

$$\frac{dx}{dt} = x(1 + x^2) \quad = f(x, y)$$

$$\frac{dy}{dt} = 3y(1 - y - x) \quad = g(x, y)$$

Find the equilibria and determine their types.

$f(x,y) = 0 \quad \Rightarrow \quad x = 0$

$g(x,y) = 0 \quad \Rightarrow \quad 3y(1-y) = 0$

$y = 0, \; y = 1$

eq. points are $(0,0)$, $(0,1)$

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1+3x^2 & 0 \\ -3y & 3-6y-3x \end{pmatrix}$$

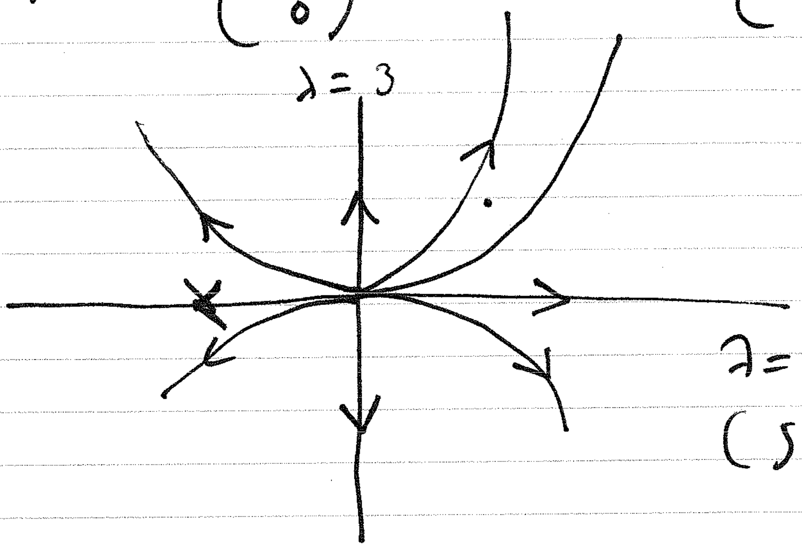
$$J_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad J_{(0,1)} = \begin{pmatrix} 1 & 0 \\ -3 & -3 \end{pmatrix}$$

Sketch linear system corresponding to the matrices above

$J_{(0,0)}$ eigenvalues / eigenvectors $\quad 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$1 \Leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 3 \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

~~$J_{(0,1)}$~~



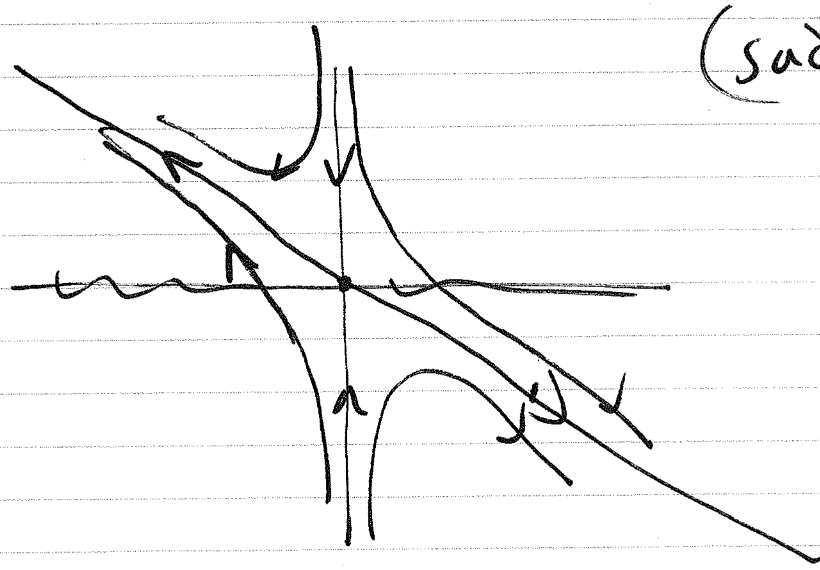
$\lambda = 1$

(source)

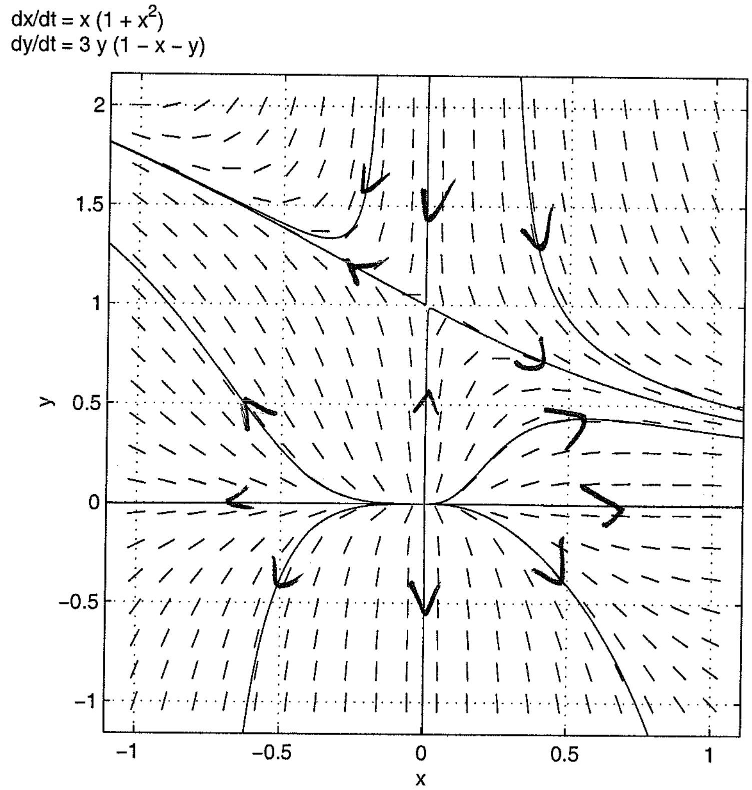
$J_{(0,1)}$ e-value evectors

$$1 \Leftrightarrow \begin{pmatrix} -4 \\ 3 \end{pmatrix} \qquad -3 \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(saddle)



The phase portrait for this system, drawn with *pplane*, is given below. Note the source at $(0,0)$ and the saddle at $(0,1)$ as predicted by our calculations.



Maths 260 Lecture 26

Topics for today

More on linearisation in nonlinear systems

Nullclines

Reading for this lecture

BDH Sections 5.1, 5.2

Suggested exercises

BDH Section 5.2; 1, 5, 7, 9, 11

Reading for next lecture

BDH Section 5.2

Today's handouts

Lecture 25 notes

Tutorial 8 questions

Classification of equilibria in nonlinear systems

Consider a nonlinear system with an equilibrium solution.

1. The equilibrium is a **sink** if all solutions that start close to the equilibrium stay close to the equilibrium for all time and tend to the equilibrium as t increases.

2. The equilibrium is a **source** if all solutions that start close to the equilibrium move away from the equilibrium as t increases.

3. The equilibrium is a **saddle** if there are curves of solutions that tend towards the equilibrium as t increases and curves of solutions that tend towards the equilibrium solution as t decreases. All other solutions started near the equilibrium move away from the equilibrium as t increases and decreases.

defn for nonlinear

To determine the type of an equilibrium in a nonlinear system, can sometimes use linearisation, i.e., use a linear system to approximate the behaviour of solutions near an equilibrium in a nonlinear system.

For most systems, knowledge of behaviour of solutions in the linearised system is sufficient to determine the behaviour near the corresponding equilibrium in the nonlinear system.

In particular, for the system

$$\frac{d\mathbf{Y}}{dt} = f(\mathbf{Y})$$

$Y = \begin{pmatrix} x \\ y \end{pmatrix}$

$f(Y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$

with an equilibrium $\mathbf{Y}(t) = \mathbf{Y}_0$, construct the linearised system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}(\mathbf{Y}_0)\mathbf{Y}$$

$\left(Y_0 = (x_0, y_0) \right)$

where $\mathbf{J}(\mathbf{Y}_0)$ is the Jacobian matrix of partial derivatives evaluated at $\mathbf{Y}_0$.

If in the linearised system the equilibrium at the origin is a sink, source, or saddle, then $\mathbf{Y} = \mathbf{Y}_0$ is a sink, source, or saddle (respectively) in the nonlinear system.

Sink: the real parts of all eigenvalues are negative.
Source: the real parts of all eigenvalues are positive.
Saddle: some real parts are positive, others negative.
Spiral: some eigenvalues are complex with non-zero real part.
Center: the eigenvalues are purely imaginary.

Note 1: A spiral is always also a saddle, source or sink.

Note 2: linearisation does not tell us anything about the behaviour of solutions to a nonlinear system far from an equilibrium.

Unfortunately, linearisation does not always work.

In particular, if the Jacobian matrix has a **zero** eigenvalue or a **purely imaginary** eigenvalue, then we cannot predict the behaviour in the nonlinear system based on linearisation alone.

Example:

$$\frac{dx}{dt} = -x^3 \qquad = f$$
$$\frac{dy}{dt} = -y + y^2 \quad = g$$

Equilibria: $\left.\begin{array}{l} -x^3 = 0 \\ -y + y^2 = 0 \end{array}\right\} \Rightarrow (0,0), (0,1)$

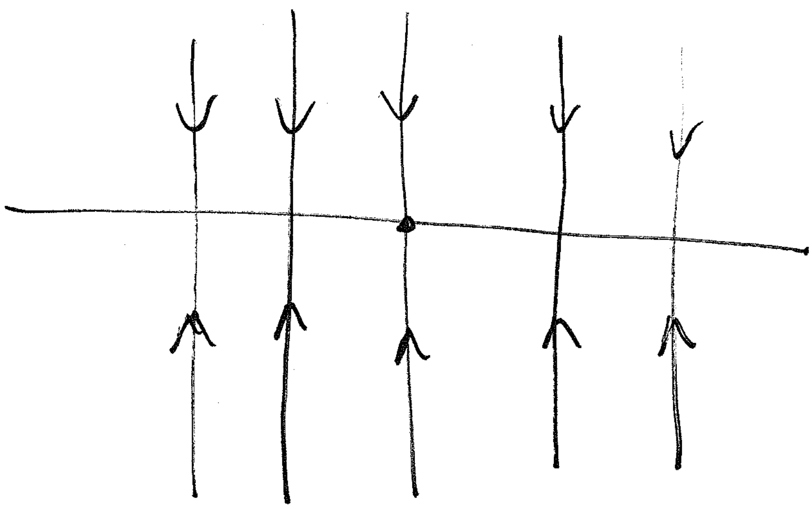
Jacobian: $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -3x^2 & 0 \\ 0 & -1+2y \end{pmatrix}$

so at $(0,0)$ the Jacobian is:

$$J_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

e-vector $\Leftrightarrow$ evalue $\quad 0 \Leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad -1 \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Linearised system has phase portrait

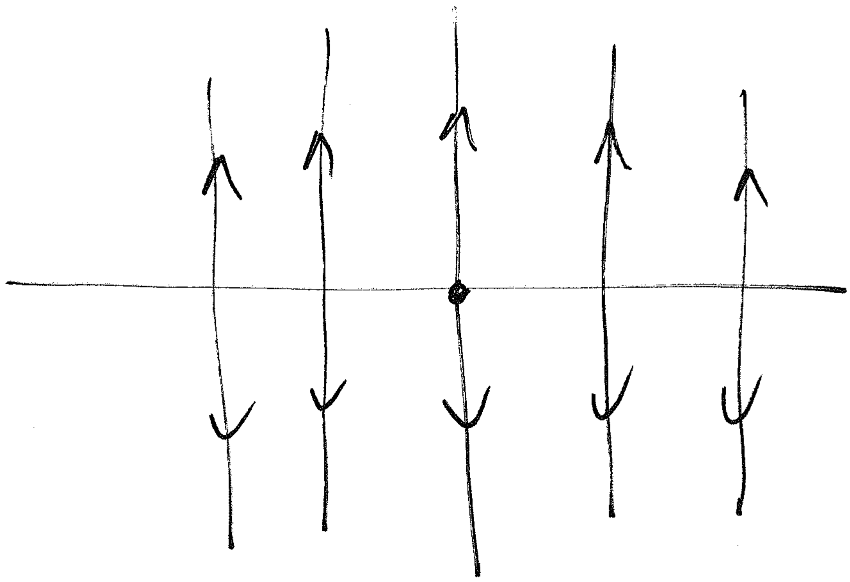


At $(0,1)$ Jacobian is:

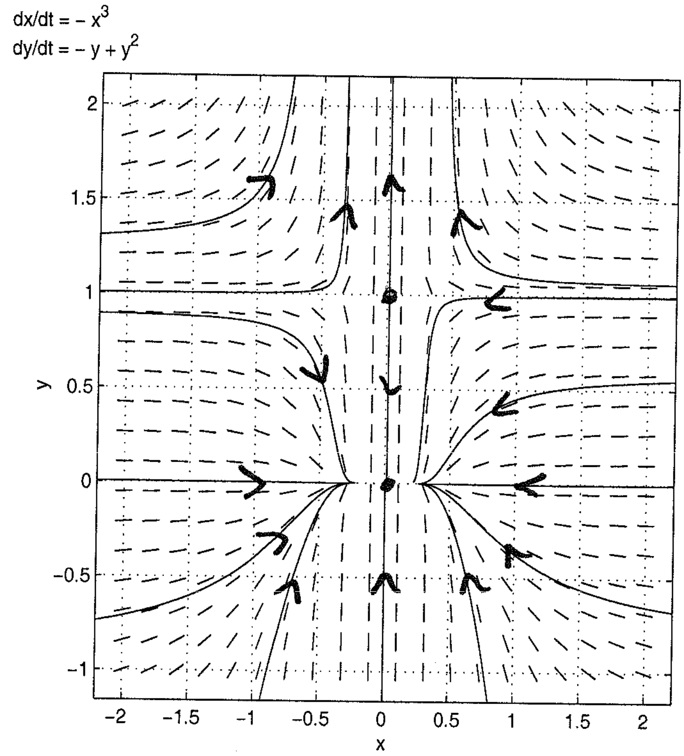
$$J_{(0,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

e-val $\leftrightarrow$ e-vec. $\quad 0 \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad 1 \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Linearised system has phase portrait



However, phase portrait for the nonlinear system is



Notice that in this phase portrait, $(0, 0)$ looks like a sink and $(0, 1)$ looks like a saddle. These results were not predicted by the corresponding linearised systems. Linearisation does not work in these cases because of the zero eigenvalues of the Jacobians.

Sketching phase portraits for nonlinear systems

We would like to be able to sketch the complete phase portrait for a nonlinear system.

Linearisation gives us good information about the behaviour of solutions near most equilibria. Can use numerics to fill in the gaps – but it would be helpful to know in advance which regions of the phase space to look at numerically.

In order to do this we can use **nullclines**.

Definition: Nullclines

Consider a system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

For this system, the x**-nullcline** is the set of points (x, y) where $f(x, y) = 0$.

The y**-nullcline** is the set of points (x, y) where $g(x, y) = 0$.

On the x-nullcline, $\frac{dx}{dt} = 0$ and the vector field is vertical, pointing straight up or straight down.

On the y-nullcline, $\frac{dy}{dt} = 0$ and the vector field is horizontal, pointing either left or right.

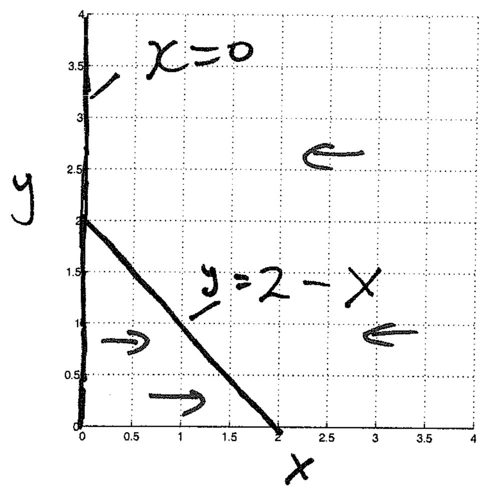
At the intersections of the x- and y-nullclines, $f(x, y) = g(x, y) = 0$, i.e., a point of intersection between an x-nullcline and a y-nullcline is an equilibrium.

Example:

Use nullclines to sketch the phase portrait for the system

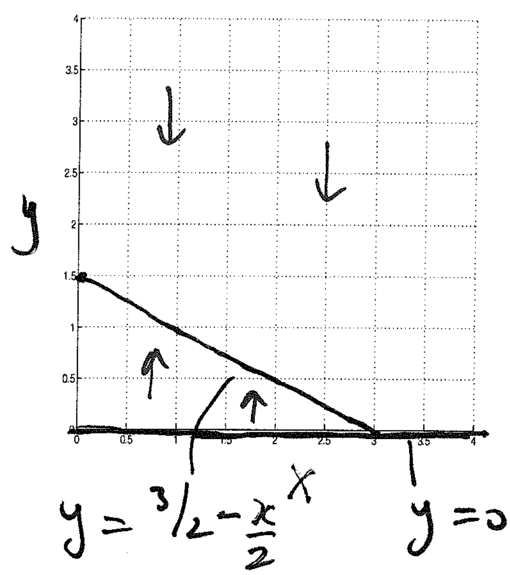
$$\frac{dx}{dt} = x(2 - x - y), \quad = f$$
$$\frac{dy}{dt} = y(3 - 2y - x), \quad = g \quad x, y \geq 0$$

The x-nullclines are:



$$f = 0 \Rightarrow \left.\begin{array}{l} x = 0 \\ 2 - x - y = 0 \end{array}\right\}$$
$$\Rightarrow \begin{array}{l} x = 0 \\ y = 2 - x \end{array}$$

The y-nullclines are:

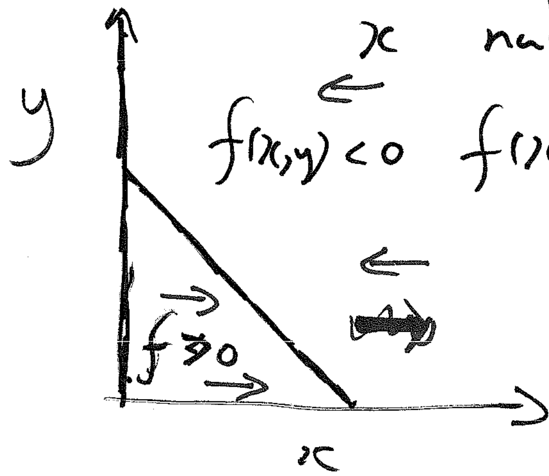


$$g = 0 \Rightarrow \left.\begin{array}{l} y = 0 \\ 3 - 2y - x = 0 \end{array}\right\}$$
$$\Rightarrow \begin{array}{l} y = 0 \\ y = \frac{3}{2} - \frac{x}{2} \end{array}$$

Nullclines divide the phase plane into regions where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ have constant sign.

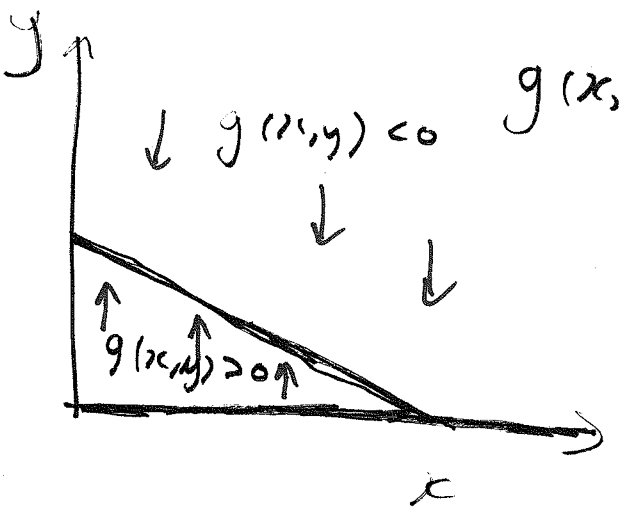
In example above:

x nullclines



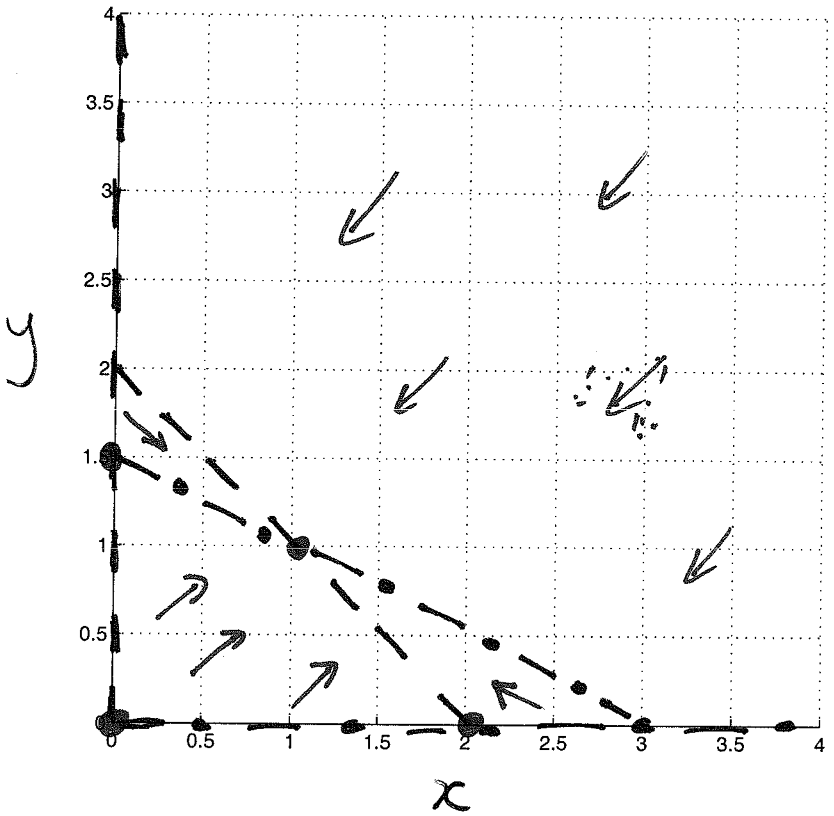
$f(x,y) = x(2-x-y)$

y nullcline



$g(x,y) = y(3-2y-x)$

Combining information about x- and y-nullclines for this example, we get



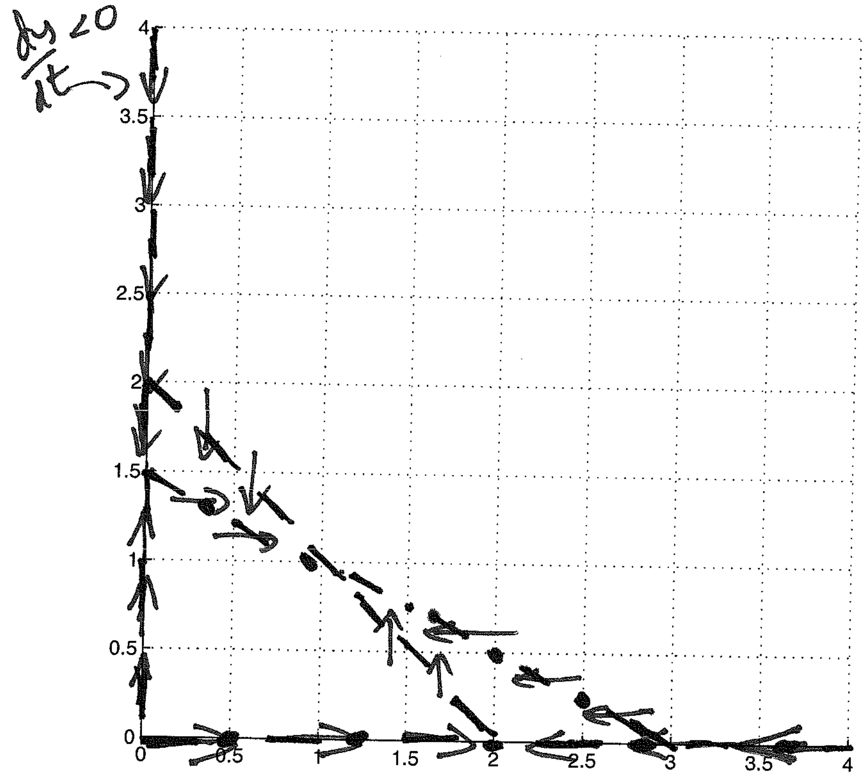
– – – x-nulcline

–•–• y-nulcline

(equilibrium points when x & y nulcline cross)

Now the phase plane is divided into four regions:

We use the known direction of solution curves in each region to determine the direction of solutions on the nullclines.



(on x - nulcline $\frac{dx}{dt} = 0$

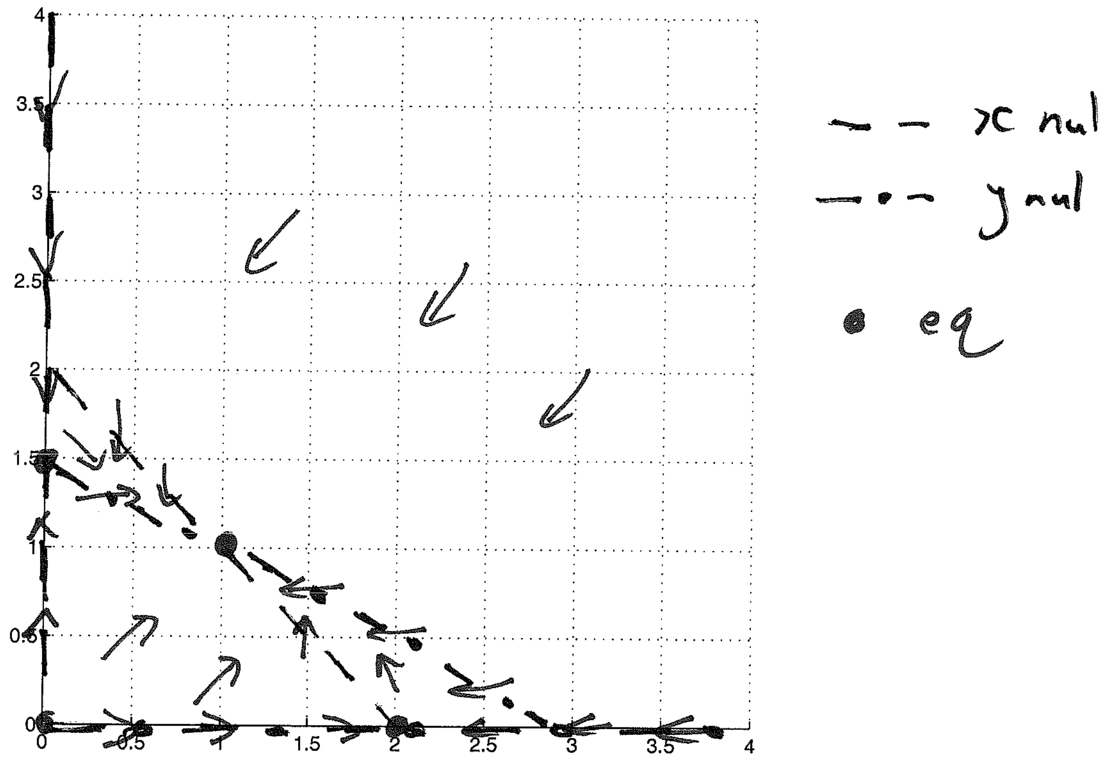
so only have vertical up or vertical down

likewise for y - nulcline.)

Now we can see that:

1. Once solutions get into region B they cannot get out again. Solutions move down and right until they get to lower right corner (i.e., equilibrium at $(1, 1)$).
2. Similarly, once solutions get into region D they cannot get out again. Solutions move up and left until they get to upper left corner (i.e., equilibrium at $(1, 1)$).
3. Solutions starting in region A or C must either leave the region by entering B or D (and then tend to $(1, 1)$) or must tend directly to $(1, 1)$.

Hence, the phase portrait for the system must be, approximately:



This picture suggests that $(1,1)$ is a sink, $(0,0)$ is a source, $(2,0)$ and $(0,3/2)$ are saddles. Linearisation confirms that this is so.

The approximate phase portrait obtained using nullclines looks very like the phase portrait obtained with *pplane*.