Maths 260 Lecture 16

Topics for today

Solutions to some special systems
Linear systems - some properties

Reading for this lecture

BDH Section 2.3, pp 175–178 (1st ed)
185–188 (2nd ed); Section 3.1

Suggested exercises

BDH Section 2.3: 5, 7, 9;
Section 3.1, 5, 7, 9, 24, 27, 29

Reading for next lecture

BDH Section 3.1 (again)

Today's handouts

Lecture 16 notes
Tutorial 5

Section 2.4 Analytic methods for some special systems

Some very special systems of DEs decouple, i.e., the rate of change of one or more of the dependent variables depends only on its own value.

Example 1

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -2y \end{aligned}$$

Example 2

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= 2x - y \end{aligned}$$

Sometimes can find analytic solutions to systems that decouple.

Example 1 again: Wish to find and plot solutions to:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -2y \end{aligned}$$

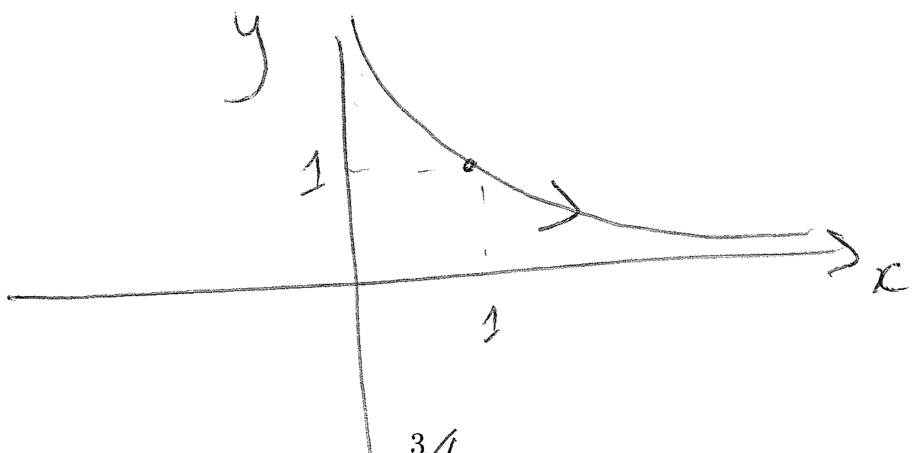
Can solve each equation separately. Find that $(x(t), y(t)) = (c_1 e^t, c_2 e^{-2t})$ is a solution for all choices of c_1 and c_2.

The values of c_1 and c_2 are determined by the initial conditions: $c_1 = x(0)$ and $c_2 = y(0)$.

Plotting solutions: e.g., $x(0) = 1$, $y(0) = 1$,

$$(x(t), y(t)) = (e^t, e^{-2t})$$

note that $x = \frac{1}{y^2}$

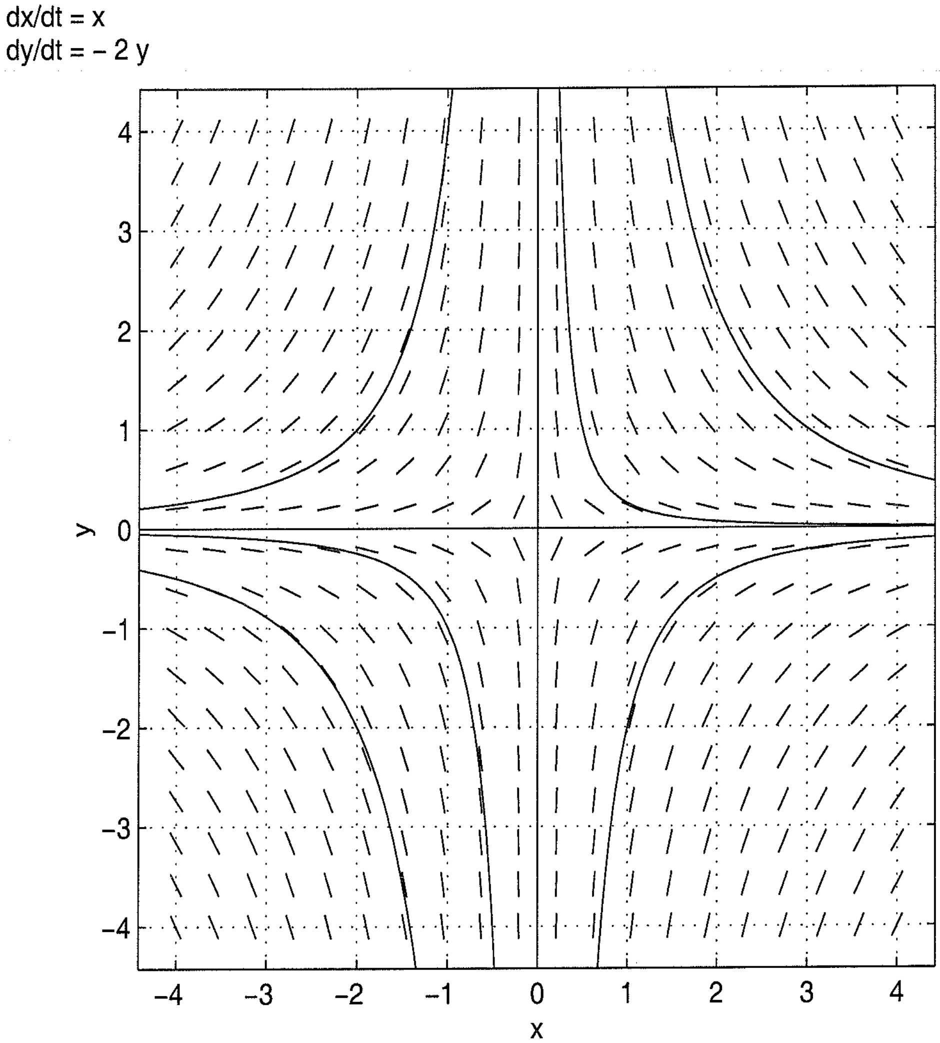


Plotting in phase space: – see previous.

Plotting some other solutions

$x(0)$	$y(0)$	c_1	c_2	$x(t)$	$y(t)$
1	-1	1	-1	e^t	$-e^{-2t}$
-2	1	-2	1	$-2e^t$	e^{-2t}
0	1	0	1	0	e^{-2t}
1	0	1	0	e^t	0

Compare phase plane plot above with numerical solutions from Matlab:



Example 2 again: Wish to find and plot solutions to:

$$\dot{x} = x,$$
$$\dot{y} = 2x - y$$

Solve first equation to get $x = c_1 e^t$. Then substitute this expression for x into second equation to get

$$\dot{y} = 2c_1 e^t - y$$

this is linear

$$\frac{dy}{dt} + y = 2c_1 e^t$$

$$\frac{d}{dt}(ye^t) = 2c_1 e^{2t}$$

$$ye^t = c_1 e^{2t} + c_2$$

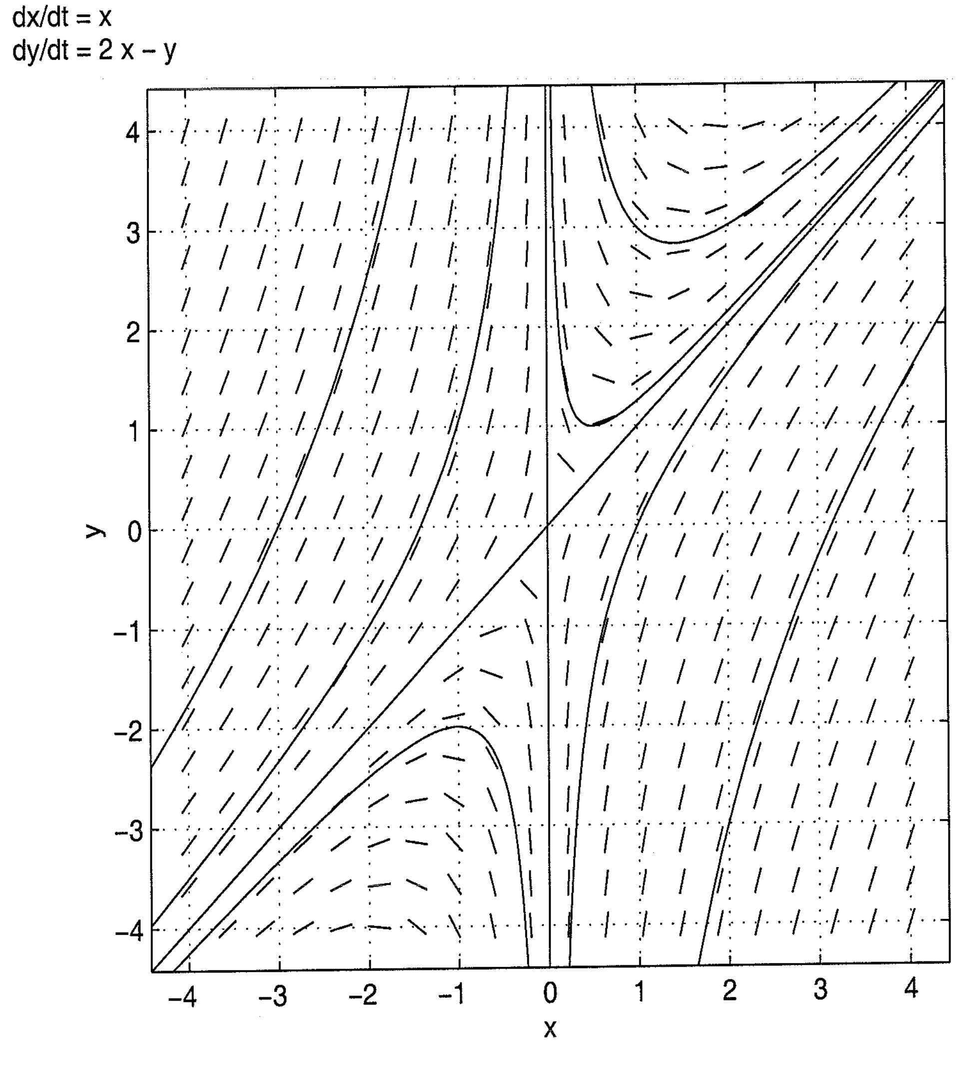
$$y = c_1 e^t + c_2 e^{-t}$$

So $(x(t), y(t)) = (c_1 e^t, c_1 e^t + c_2 e^{-t})$ is a solution for all choices of c_1 and c_2.

Plotting solutions in phase space

$x(0)$	$y(0)$	c_1	c_2	$x(t)$	$y(t)$
1	1	1	0	e^t	e^t
0	1	0	1	0	e^{-t}
1	0	1	-1	e^t	$e^t - e^{-t}$
0.25	1	$1/4$	$3/4$	$1/4 e^t$	$1/4 e^t + 3/4 e^{-t}$

Compare phase plane plot above with numerical solutions from Matlab: In both



examples, system could be solved. Found that some solutions gave straightline solution curves in phase portrait but (from Matlab) most solution curves not straight lines.

Section 2.5 Linear Systems

Linear systems are an important class of systems of DEs, partly because some important models are linear but also because we can use linear systems to help understand nonlinear systems.

A linear system is a system of DEs where the dependent variables only appear to the first power.

Mostly interested in systems that can be written as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

where $\mathbf{Y}$ is a vector:

$$\mathbf{Y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

and $\mathbf{A}$ is a matrix of constants:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

Example

$$\frac{dx}{dt} = 2x - z$$
$$\frac{dy}{dt} = -x - z$$
$$\frac{dz}{dt} = x + y$$

Can write this system as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{Y} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

A is called the coefficient matrix. The number of dependent variables is called the dimension of the system.

Some properties of linear systems

1. Equilibrium solutions

Want to find equilibrium solutions of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

i.e., find $\mathbf{Y_0}$ such that $\mathbf{AY_0} = \mathbf{0}$.

From linear algebra, know that if $\det(\mathbf{A}) \neq 0$, then the only solution of $\mathbf{AY_0} = \mathbf{0}$ is $\mathbf{Y_0} = \mathbf{0}$ (called the trivial solution).

Thus, if $\det(\mathbf{A}) \neq 0$, then $\mathbf{Y}(t) = \mathbf{0}$ is the only equilibrium solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

2. Linearity Principle

If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are both solutions to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

then so is

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

for any constants k_1 and k_2.

The function

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

is called a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

Example 2 again

$$\dot{x} = x,$$
$$\dot{y} = 2x - y$$

Found earlier that

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad \mathbf{Y}_2(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

are (straight line) solutions of this system and that all solutions can be written as

$$\mathbf{Y}(t) = \begin{pmatrix} c_1 e^t \\ c_1 e^t + c_2 e^{-t} \end{pmatrix} = c_1 \mathbf{Y}_1 + c_2 \mathbf{Y_2},$$

i.e., as a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

Important ideas for today

Solutions for systems that decouple

Straight line solutions

Linear systems - linearity principle

Maths 260 Lecture 18

Topic for today

Straight-line solutions

Reading for this lecture

BDH Section 3.2

Suggested exercises

BDH Section 3.2, 1, 5, 11, 13, 25

Reading for next lecture

BDH Section 3.3

Today's handouts

Lecture 18 notes

Assignment 3

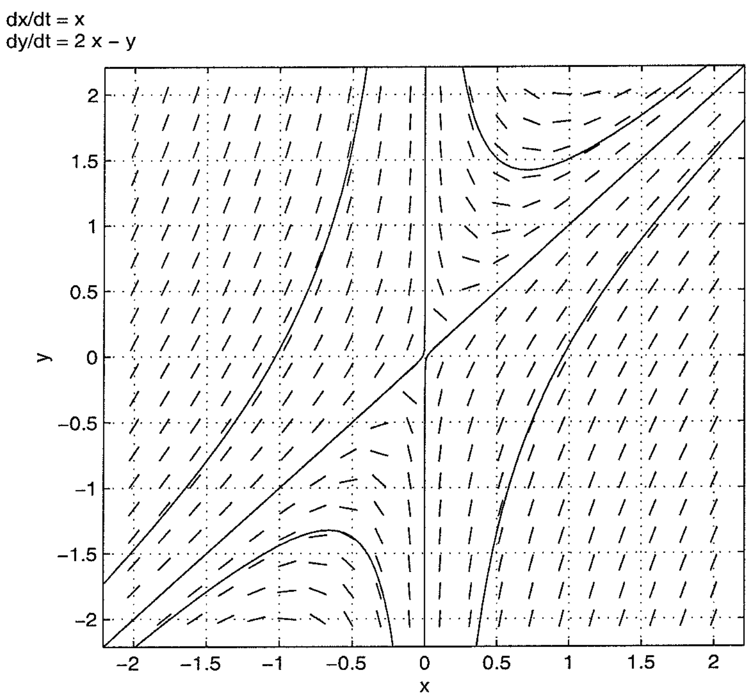
Section 2.6 Straight-Line Solutions

Given a system of DEs such as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix},$$

we want to find two linearly independent solution vectors that could be used to construct the general solution.

Direction field and some solutions:



Note straight-line solutions - these are linearly independent solutions. Can we find them?

To find a straight-line solution, note that at a point (x, y) on a straight-line solution, the vector field at that point must point in the same (or opposite) direction as the vector from the origin to (x, y).

This means

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

where $\mathbf{v} = (x, y)$ and λ is a real number.

If $\lambda > 0$, vector field points in same direction as $\mathbf{v}$, i.e., away from the origin.

If $\lambda < 0$, vector field points in opposite direction to $\mathbf{v}$, i.e., towards the origin.

Equation (1) says that $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ.

Claim: to get a formula for the straight-line solutions, write

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$$

where $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ.

Then $\mathbf{Y}(t)$ is a solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

$\Rightarrow \quad \lambda e^{\lambda t} V = e^{\lambda t} AV$

now $\quad e^{\lambda t} \neq 0$

$\Rightarrow \quad \lambda V = AV$

Also, as t varies, $e^{\lambda t}$ just increases or decreases or remains constant (depending on λ) and $\mathbf{v}$ is constant so the solution curve for $\mathbf{Y}(t)$ is a straight line.

Summary

Can find a straight-line solution of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

by finding a real eigenvalue, λ, of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$; a straight-line solution is then

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$$

Example

Find any straight-line solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

solve $\lambda v = Av$

$\Rightarrow \left(A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\vec{v} = 0$

just equation for eigenvalue

$\det \begin{pmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{pmatrix} = 0$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

solve $\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}\vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

& $\begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}\vec{v} = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Note: In this example the two straight-line solutions are linearly independent. This is as expected because:

Result from linear algebra

Let λ_1 and λ_2 be two real and distinct eigenvalues for the matrix $\mathbf{A}$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Hence, the two straight-line solutions $\mathbf{Y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{Y}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$ are linearly independent at $t = 0$ and thus are linearly independent solutions.

Grand summary

If $\mathbf{A}$ is an $m \times m$ matrix with real eigenvalues

$$\lambda_1, ..., \lambda_k$$

and corresponding eigenvectors

$$\mathbf{v}_1, ..., \mathbf{v}_k,$$

then

$$\mathbf{Y}_1 = e^{\lambda_1 t}\mathbf{v}_1, \ ..., \mathbf{Y}_k = e^{\lambda_k t}\mathbf{v}_k$$

are straight-line solutions of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

Furthermore, if all the λ_i are distinct and $k = m$ (i.e., there are m real and distinct eigenvalues of A), then the set $\{\mathbf{Y}_1, ..., \mathbf{Y}_k\}$ is linearly independent and the general solution to the system is

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1 + ... + c_m\mathbf{Y}_m.$$

Examples

1. Find the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \mathbf{A} = \begin{pmatrix} -3 & -1 \\ -2 & -4 \end{pmatrix}.$$

Describe the longterm behaviour of solutions.

Find eigenvalues

$$\det \begin{pmatrix} -3-\lambda & -1 \\ -2 & -4-\lambda \end{pmatrix} = \lambda^2 + 7\lambda + 12 - 2 = 0$$

$$\lambda = -5, -2$$

$$\lambda = -5 \quad \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

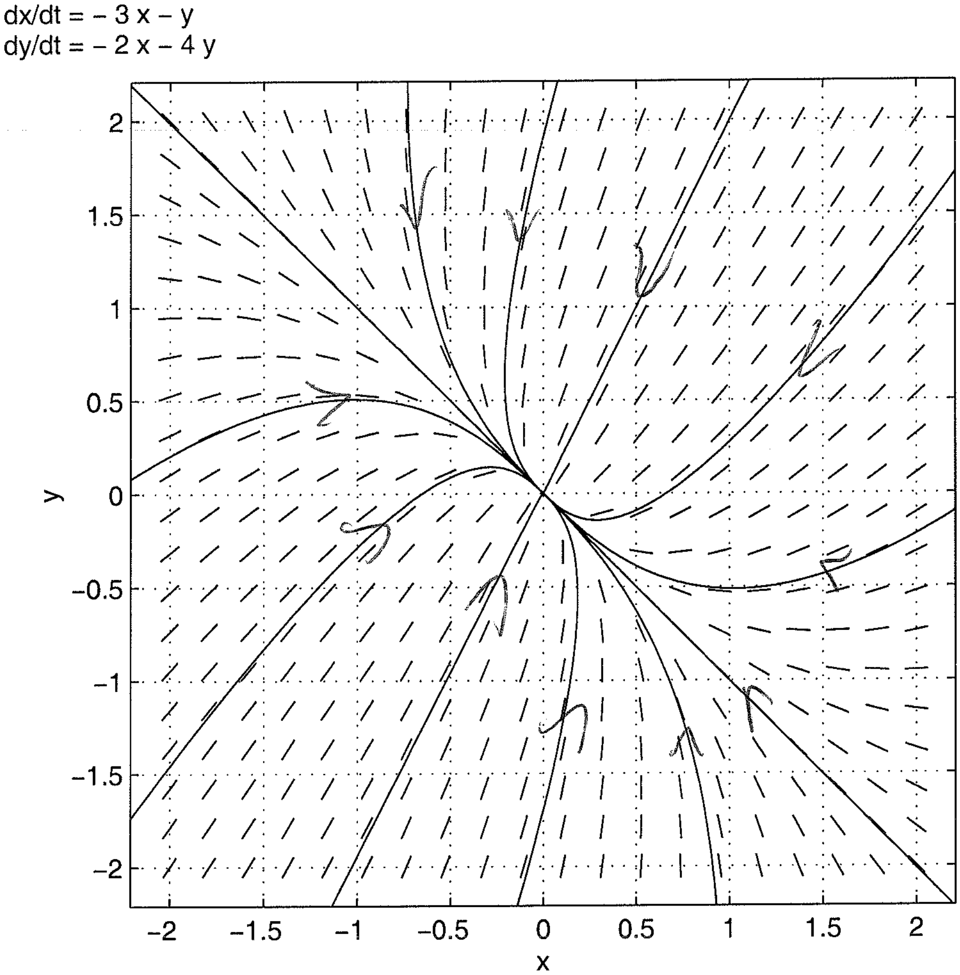
$$\lambda = -2 \quad \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \vec{v} = 0 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

general soln

$$Y = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

$Y \to 0$ as $t \to \infty$

Direction field and some solutions



2. Find the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{A} = \begin{pmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Describe the longterm behaviour of solutions.

eigenvalue / eigenvectors

$\lambda = -5 \quad \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}, \quad \lambda = -1 \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1 \quad \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$Y = c_1 \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{t}$$

$Y(t) \to \infty$ along $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ unless $c_3 = 0$

Maths 260 Lecture 17

Topics for today

Linear independence of vectors and solutions
The general solution

Reading for this lecture

BDH Section 3.1

Suggested exercises

BDH Section 3.1, 31, 33, 34, 35

Reading for next lecture

BDH Section 3.2

Today's handouts

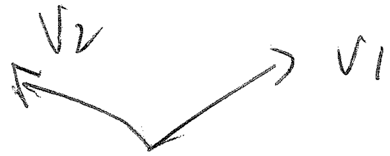
Lecture 17 notes

Section 2.5 Linear Systems (continued)

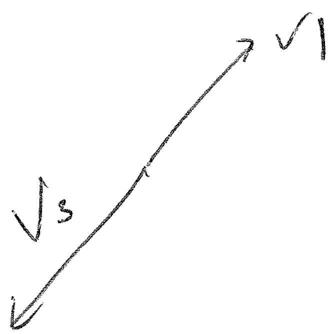
Linear Independence of Vectors

Two vectors in the plane are **linearly independent** if neither vector is a multiple of the other, i.e., if they do not both lie on the same line through the origin.

e.g. $v_1 = (1, 1)$, $v_2 = (2, -1)$ are linearly independent.



e.g. $v_1 = (1, 1)$ and $v_3 = (-2, -2)$ are linearly dependent.



Important result: If two vectors (x_1, y_1) and (x_2, y_2) are linearly independent vectors in the plane, then for any other vector (x_0, y_0) there exists k_1, k_2 such that

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Important consequence of this result:

Consider the DE

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ is a 2×2 matrix.

If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions to the system with $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ being linearly independent vectors, then for any initial condition

$$\mathbf{Y}(0) = (x_0, y_0)$$

we can find constants k_1 and k_2 so that $k_1\mathbf{Y_1}(t) + k_2\mathbf{Y_1}(t)$ is the solution to IVP

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

i.e., every solution can be expressed as a linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

Note: If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions with $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ linearly independent, then $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent vectors for all t.

In this case we say that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent solutions.

Summary: For the system

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$

if we can find two linearly independent solutions we can write down the general solution and hence solve any IVP arising from this DE.

These results can be generalised to higher dimensions: A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ is **linearly dependent** if there are constants $c_1, c_2, \ldots, c_m$ (not all zero) such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_m\mathbf{v_m} = \mathbf{0} \qquad (1)$$

If all the c_i are zero whenever equation (1) is satisfied, the set of vectors is linearly independent.

Checking linear independence of solution vectors in higher dimensions

(This method works in two dimensions also, but the earlier method is quicker in this case.)

If $\mathbf{Y}_1(t), \mathbf{Y}_2(t), ..., \mathbf{Y}_m(t)$ are solutions for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ is an $m \times m$ matrix, then the set of solution vectors $\{\mathbf{Y_1}, ..., \mathbf{Y_m}\}$ is linearly independent if and only if

$$W(\mathbf{Y}_1, ..., \mathbf{Y}_m)(t) = \det(\mathbf{Y}_1 \; ... \; \mathbf{Y}_m) \neq 0$$

for all t.

Example: The vectors

$$\begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{5t}, \quad \begin{pmatrix} 2t + \frac{1}{2} \\ -\frac{1}{2} \\ -t - 1 \end{pmatrix} e^{5t}$$

are all solution vectors to some system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

Are they linearly independent?

$$W = \det \begin{pmatrix} -4 & 2e^{5t} & (2t+1/2)e^{5t} \\ -5 & 0 & -1/2\, e^{5t} \\ 2 & -e^{+5t} & (-t-1)e^{5t} \end{pmatrix}$$

$$= -2e^{5t} \begin{vmatrix} -5 & 1/2\, e^{5t} \\ 2 & (-t-1)e^{5t} \end{vmatrix}$$

$$+ e^{5t} \begin{vmatrix} 4 & 2t+1/2 \\ -5 & -1/2\, e^{5t} \end{vmatrix}$$

$$= e^{10t} \left(-2(5t+5-1) - 2 + 10t - 5/2 \right)$$

$$= e^{10t} \left(-8 - 2 - 5/2 \right)$$

Notes:

1. This test (the **Wronskian** test) does not work if the vectors are not all solution vectors for the same system.
2. It turns out that the Wronskian is either identically zero (i.e., $W(t) = 0$ for all t) or $W(t) \neq 0$ for any t. Therefore, only need to calculate the Wronskian at one value of t, say $t = 0$.
3. We need m solution vectors to do the test.

Main result: If $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$,..., $\mathbf{Y}_m(t)$, are linearly independent solution vectors to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

where $\mathbf{A}$ is an $m \times m$ matrix, then the general solution to the system is

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1(t) + c_2\mathbf{Y}_2(t) + \cdots + c_m\mathbf{Y}_m(t)$$

where c_1, c_2,..., c_m are arbitrary constants. That is, every solution to the system can be written in this form by appropriate choice of c_1, c_2,..., c_m.

Example: Show that the vectors

$$e^{2t}\begin{pmatrix}8\\4\\1\end{pmatrix},\quad e^{-4t}\begin{pmatrix}0\\-2\\1\end{pmatrix},\quad e^{-2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

are linearly independent solutions to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{A} = \begin{pmatrix}2 & 0 & 0\\3 & -4 & 0\\0 & 1 & -2\end{pmatrix}.$$

Hence find the solution to the IVP

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix}0\\2\\0\end{pmatrix}.$$

$$W = \det\left(e^{-4t}\begin{pmatrix}8 & 0 & 0\\4 & -2 & 0\\1 & 1 & 1\end{pmatrix}\right) = -16e^{-4t}$$

$$Y = c_1\begin{pmatrix}8\\4\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}0\\-2\\1\end{pmatrix}e^{-4t} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-2t}$$

$$Y(0) = \begin{pmatrix}0\\2\\0\end{pmatrix} \Rightarrow c_2 = -1, \quad c_3 = 1$$

$$Y = \begin{pmatrix}0\\2\\-1\end{pmatrix}e^{-4t} + \begin{pmatrix}0\\0\\1\end{pmatrix}e^{-2t}$$