Maths 260 Lecture 19

Topic for today

Classification of equilibria in linear systems with real eigenvalues

Reading for this lecture

BDH Section 3.3

Suggested exercises

BDH Section 3.3, 1, 5, 9, 11, 19

Reading for next lecture

The handout on complex numbers

Today's handouts

Lecture 19 notes,
The handout on complex numbers

Section 2.7 Classification of equilibria in linear systems with real eigenvalues

This lecture looks at systems of the form

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a matrix with real eigenvalues only.

All such systems have an equilibrium at the origin: we are interested in the behaviour of solutions near the origin, especially when viewed in phase space.

Example Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues of coefficient matrix are $\lambda = 3, -4$ with eigenvectors

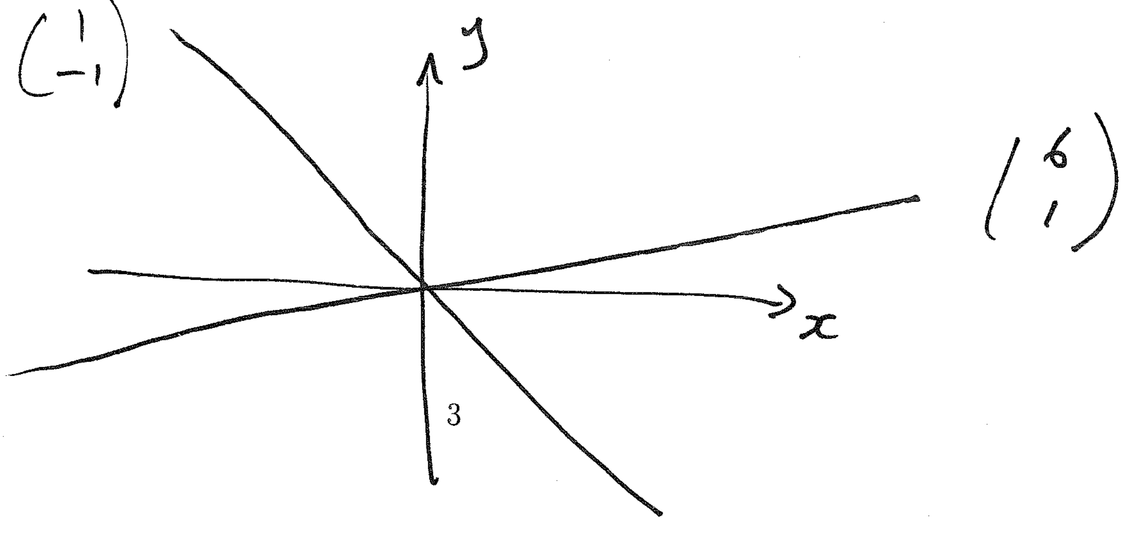
$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively.

The general solution is:

$$Y(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Straight-line solutions are:



To see behaviour of solutions that are not straight-line solutions, i.e., solutions with $c_1 \neq 0$ and $c_2 \neq 0$, note that as $t \to \infty$

$$\mathbf{Y}(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

i.e., as $t \to \infty$, these solutions behave like the straight-line solution

$$c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

Similarly, as $t \to -\infty$, these solutions behave like the straight-line solution

$$c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This example illustrates typical behaviour of solutions to a planar linear system with one positive real eigenvalue and one negative real eigenvalue.

A characteristic feature of phase portrait is the presence of two special lines:

- On one line, solutions tend to origin as $t \to \infty$;
- On other line, solutions tend to origin as $t \to -\infty$.
- All other solutions tend to ∞ as $t \to \pm\infty$.

The equilibrium point at the origin in this type of system is called a **saddle**.

Example Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues of coefficient matrix are $\lambda = -5, -2$ with eigenvectors

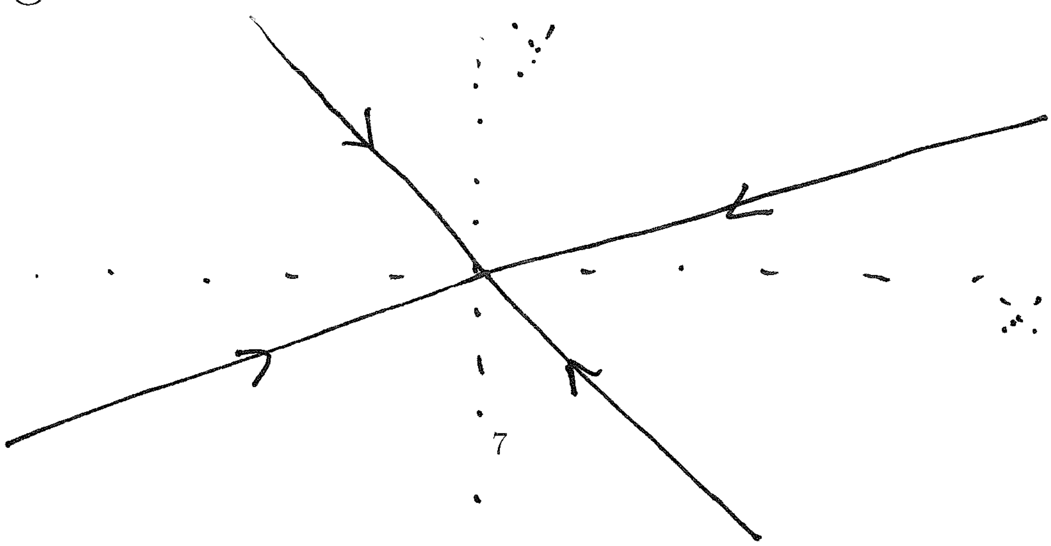
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively.

The general solution is:

$$Y = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

Straight-line solutions are:



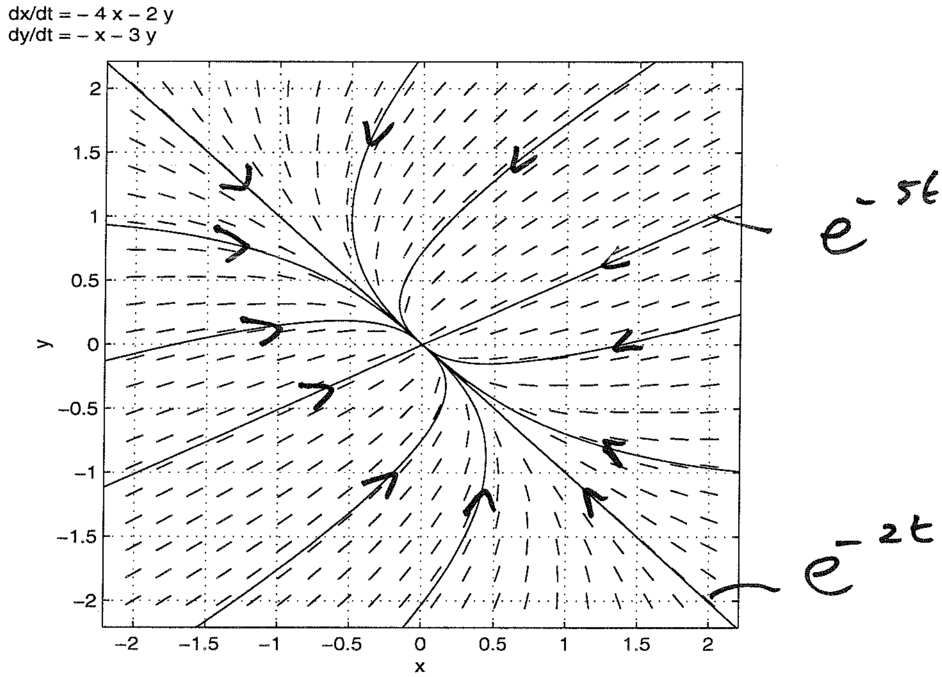
As $t \to \infty$, $e^{-5t} \to 0$ and $e^{-2t} \to 0$, so all solutions tend to the origin as $t \to \infty$.

This is a general result: if all eigenvalues of matrix $\mathbf{A}$ are real and negative, then all solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

tend to the origin as $t \to \infty$.

Direction field and some solutions:



This picture suggests that most solutions are tangent to the straight line solution

$$e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as $t \to -\infty$. We can prove this:

Slope of solution curves is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$Y = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

(from soln)

$$\Rightarrow \quad x(t) = 2c_1 e^{-5t} + c_2 e^{-2t}$$

$$y(t) = c_1 e^{-5t} - c_2 e^{-2t}$$

$$\frac{dx}{dt} = -10 c_1 e^{-5t} - 2 c_2 e^{-2t}$$

$$\frac{dy}{dt} = -5 c_1 e^{-5t} + 2 c_2 e^{-2t}$$

$$\frac{dy}{dx} = \text{slope of ~~slo~~ soln}$$

$$= \frac{-5 c_1 e^{-5t} + 2 c_2 e^{-2t}}{-10 c_1 e^{-5t} - 2 c_2 e^{-2t}}$$

$$\lim_{t \to \infty} \frac{dy}{dx} = \lim_{t \to \infty} \frac{-5 c_1 e^{-3t} + 2 c_2}{-10 c_1 e^{-3t} - 2 c_2}$$

$$= \frac{2 c_2}{-2 c_2} = -1 \quad (\text{provided } c_2 \neq 0)$$

so, if $c_2 \neq 0$,

$$\lim_{t \to \infty} \left(\frac{dy}{dx} \right) = -1.$$

Thus as $t \to \infty$, all solutions tend to the origin and almost all are tangent to the straight-line solution

$$e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In general, in a linear system with 2 real negative eigenvalues, $\lambda_1 < \lambda_2 < 0$, all solutions tend to the origin as $t \to \infty$.

Except for those solutions starting on the line of eigenvectors corresponding to λ_1, all solutions are tangent at $(0, 0)$ to the line of eigenvector corresponding to λ_2.

The equilibrium point in this type of system is called a **sink**.

Example Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues of coefficient matrix are $\lambda = 5, 2$ with eigenvectors

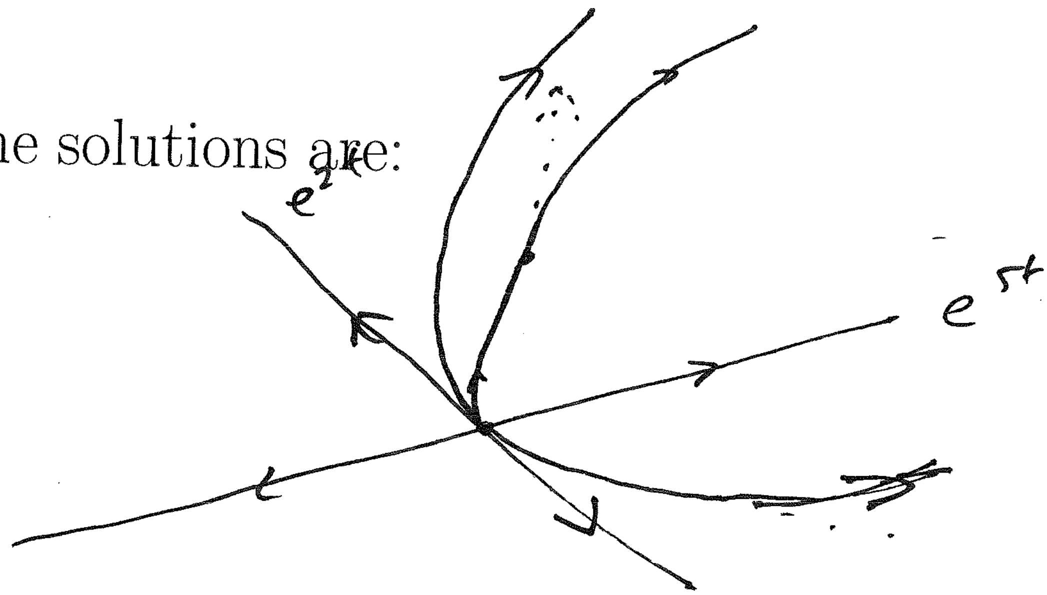
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively.

The general solution is:

$$Y = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

Straight-line solutions are:



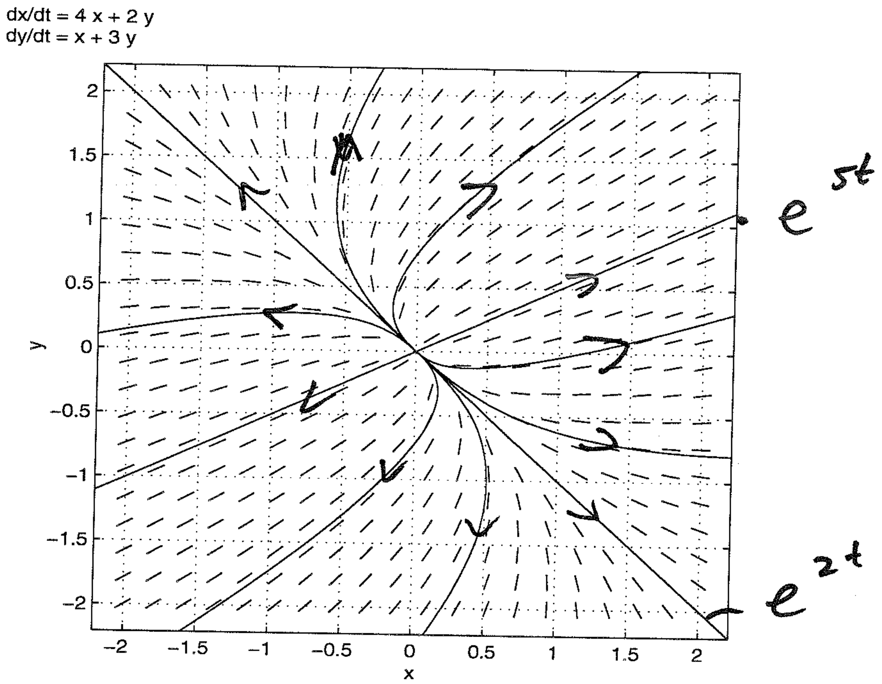
As $t \to \infty$, all non-zero solutions move away from the origin.

This is a general result: if all eigenvalues of $\mathbf{A}$ are real and positive, all non-zero solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

tend away from the origin as $t \to \infty$.

Direction field and some solutions:



This picture suggests that most solutions are tangent to the straight line solution

$$e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as $t \longrightarrow \infty$. We can prove this either

- by method used in last example, or
- by noting that this example corresponds to reversing time in the last example. Hence, phase portrait is the same as in last example but with direction of arrows reversed.

This is a general result. If $\mathbf{A}$ is a 2×2 matrix with eigenvalues λ_1 and λ_2, with $0 < \lambda_2 < \lambda_1$, then except for those solutions starting on the line of the eigenvectors corresponding to λ_1, all solutions are tangent at $(0, 0)$ to the line of eigenvectors corresponding to λ_2.

The equilibrium point in this case is called a **source**.

This classification of equilibria extends to higher dimensions:

For the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

$\mathbf{Y} = \mathbf{0}$ is always an equilibrium. Assuming that all eigenvalues of $\mathbf{A}$ are real and distinct, then:

1. If all eigenvalues of $\mathbf{A}$ are positive, $\mathbf{Y} = \mathbf{0}$ is a **source**.
2. If all eigenvalues of $\mathbf{A}$ are negative, $\mathbf{Y} = \mathbf{0}$ is a **sink**.
3. If at least one eigenvalue of $\mathbf{A}$ is negative and at least one eigenvalue is positive, $\mathbf{Y} = \mathbf{0}$ is a **saddle**.

Maths 260 Lecture 20

Topic for today

Complex Numbers

Reading for this lecture

The handout on complex numbers.

Reading for next lecture

The handout on complex numbers

Today's handouts

Lecture 20 notes

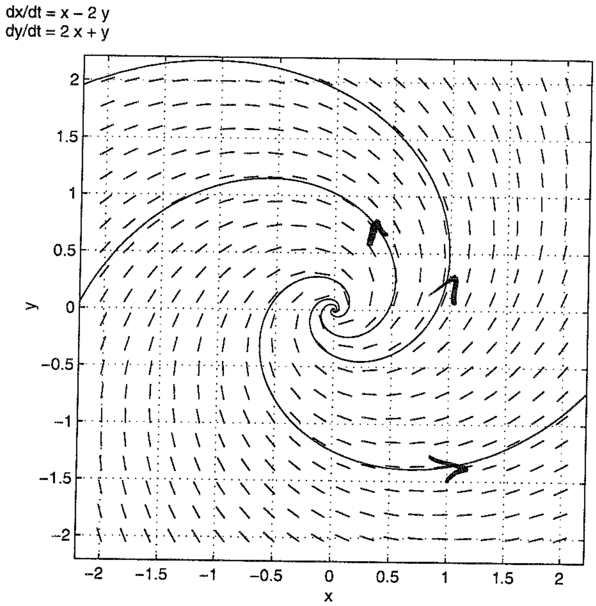
2.8.1 Complex Numbers

In order to apply analytic techniques to systems of DEs, we need to know about complex numbers and their properties.

Example Consider the system

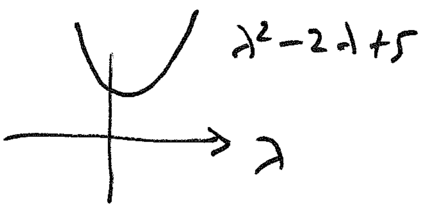
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

Slope field and some solutions



There are no straight-line solutions! Let's see if we can find out why.
Calculate the eigenvalues:

$$0 = \det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 5.$$



So the quadratic formula gives:

$$\lambda = \frac{+2 \pm \sqrt{4-20}}{2}$$
$$= 1 \pm \frac{\sqrt{-16}}{2}$$

Notes:

- We need the square root of a negative number!
- The matrix doesn't have any eigenvalues that are *real numbers*. That's why there are no straight-line solutions to the system of DEs.
- However, we can still find eigenvalues that are *complex numbers*.
- We *can* still calculate two eigenvalues provided we introduce a new number:

$$i = \sqrt{-1}$$

$$\lambda = 1 \pm \frac{\sqrt{16}\sqrt{-1}}{2} = 1 \pm \frac{4i}{2}$$
$$= 1 \pm 2i$$

Example Solve $x^2 + 2x + 5 = 0$.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= -1 \pm 2i$$

Then solutions to above equation are:

We can use i to get solutions to any quadratic equation

$$ax^2 + bx + c = 0$$

where $b^2 - 4ac < 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac < 0$

$$\Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} \cdot i$$

In fact, one can prove: **Fundamental Theorem of Algebra:**

An nth degree equation $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$ has n solutions.
This means that the polynomial can be factorised:

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$
$$= a_n(x - x_1)(x - x_2)(x - x_3) \dots (x - x_n)$$

where $x_1, x_2, \dots, x_n$ are the roots of the equation. Note that some of these x_i may be repeated.

Definition A complex number is a number of the form

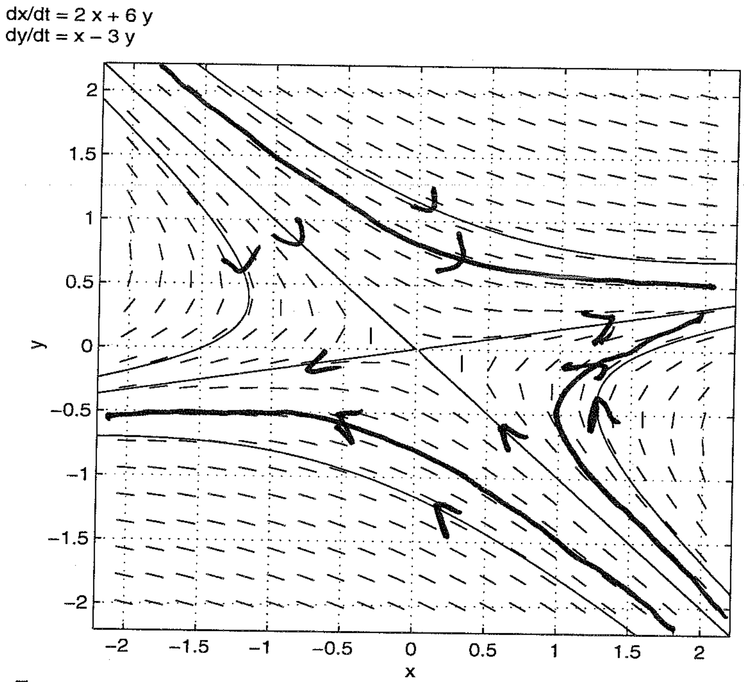
$$z = a + ib$$

where a and b are real and $i^2 = -1$.

Can think of a complex number as a *pair* of real numbers.

Geometric Interpretation - the **Argand Diagram**.

Direction field and some solutions



Note that on solution curves for straight-line solution

$$\mathbf{Y}_1(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix},$$

the arrows point away from the origin, indicating that $\mathbf{Y}_1(t) \to \mathbf{0}$ as $t \to -\infty$.

Similarly, arrows on solution curves for straight-line solution

$$\mathbf{Y}_2(t) = c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

point towards the origin, indicating that $\mathbf{Y}_2(t) \to \mathbf{0}$ as $t \to \infty$.

Definitions

- The **complex conjugate** of a complex number $z = a + ib$ is

$$\bar{z} = a - ib$$

- The **real part** of z is Re $z = a$.
- The **imaginary part** of z is Im $z = b$.

Example: $z = 2 - 3i$

$\bar{z} =$ 2 + 3i
Re $z =$ 2
Im $z =$ −3 (not −3i)

Algebra of Complex Numbers

Addition/Subtraction: collect real and imaginary terms.

Example: $(2 + 4i) + (3 - 2i) =$ 5 + 2i

Example: $(-1 - 4i) - (4 + 3i) =$ −5 − 7i

Multiplication: Multiply out brackets and collect real and imaginary terms, remembering that $i^2 = -1$.

Example: $(2+4i)(3-2i) = 2.3 + 2(-2i) + 4i.3 + 4i(-2i)$
$= 6 - 4i + 12i + 8$
$= 14 + 8i$

$(-1-4i)(4+3i) =$
$= 8 - 19i$

Division: Multiply top and bottom by complex conjugate of denominator, then collect real and imaginary terms.

Example: $\dfrac{2+4i}{3-2i} = \dfrac{2+4i}{3-2i} \cdot \dfrac{3+2i}{3+2i}$

$= \dfrac{(2+4i)(3+2i)}{(3-2i)(3+2i)} = \dfrac{-2+16i}{13}$

$= -\dfrac{2}{13} + \dfrac{16}{13} i$

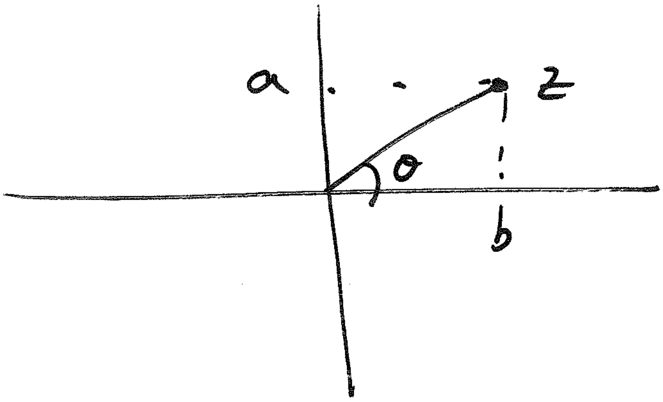
Example: $\dfrac{-1-4i}{4+3i} =$

$= \dfrac{-1-4i}{4+3i} \dfrac{4-3i}{4-3i} = \dfrac{-16-13i}{25}$

Polar Form

Another way of describing a given complex number is to use *polar co-ordinates*: the distance of

z from 0 and the angle it makes with the real axis.

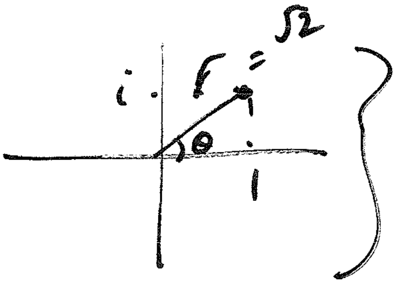


Note: $a = r\cos\theta$, $b = r\sin\theta$.

So $z = a + ib = r(\cos\theta + i\sin\theta)$.

And so $r = \sqrt{a^2 + b^2}$ (the *modulus* of z), denoted $|z|$, (the length)
and $\theta = \tan^{-1}(b/a)$ (the *argument* of z), denoted $\arg z$. (being careful about quadrant)

Example Convert $z = 1 + i$ into polar form.



$z = \sqrt{2}\cos \pi/4 + \sqrt{2}\sin \pi/4\, i$

from picture $\theta = \pi/4$
$r = \sqrt{2}$

Example Convert $z = 3(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$ into rectangular form. $= -3i$

($\cos 3\pi/2 = 0$
$\sin 3\pi/2 = -1$)