Maths 260 Lecture 1

Topics for today

Introduction to differential equations

Introduction to modelling

Reading for this lecture

BDH Section 1.1

Suggested Exercises

BDH Section 1.1: 1, 3, 13, 15

Reading for next lecture

BDH Section 1.2

Today's handouts

Course guide

Lecture 1 notes

Section 1.1 Modelling with Differential Equations

The subject of differential equations is about using derivatives to describe how a quantity changes.

Using knowledge about how a quantity changes to write down a DE is called *modelling*, and a DE is a *model.*

The goal of modelling is to use the DE model to predict future values of the quantity being modelled.

Today's class gives an overview of some types of models we look at in this course.

Important steps in making a model:

1. Identify assumptions on which the model is based.

2. Identify all relevant quantities in the model.

3. Use assumptions in (1) to write down equations relating the quantities in (2).

Quantities in a model divide into three types:

(a) independent variables

e.g. time & space
(t) (x)

i.e. $y = f(x)$ → x independent
→ y dependent

(b) dependent variables

population

amount of radioactive material

amount of wet paint

(c) parameters

→ rate of radioactive decay

→ rate paint dries

→ rate population increases

Keep the model as simple as possible!

Example 1: Single Population, Unlimited Growth

Assume: Population grows at a rate proportional to the size of the population

Quantities:
t=time (independent variable)
P=size of population (dependent variable)
k=proportionality constant (parameter) >0

Model:

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = kP, \quad k > 0 \quad (*)$$

(Note: we cannot solve $P = \int kP\,dt + c$)

Predictions of the model:

soln of (*) $P = P_0 e^{kt}$

where P_0 = initial population (at $t=0$)

population grows without limit but never becomes ∞ in finite time.
∞ = infinity

Example 2: Single Population, Limited Growth

<u>Assume</u>: If the population is small, the population grows at a rate proportional to the size of the population.

If the population is too large, the population will decrease.

<u>Quantities:</u>
t=time (independent variable)
P=size of population (dependent variable)
k=growth rate coefficient for small population
N=maximum size of population before growth negative

<u>Model</u>:

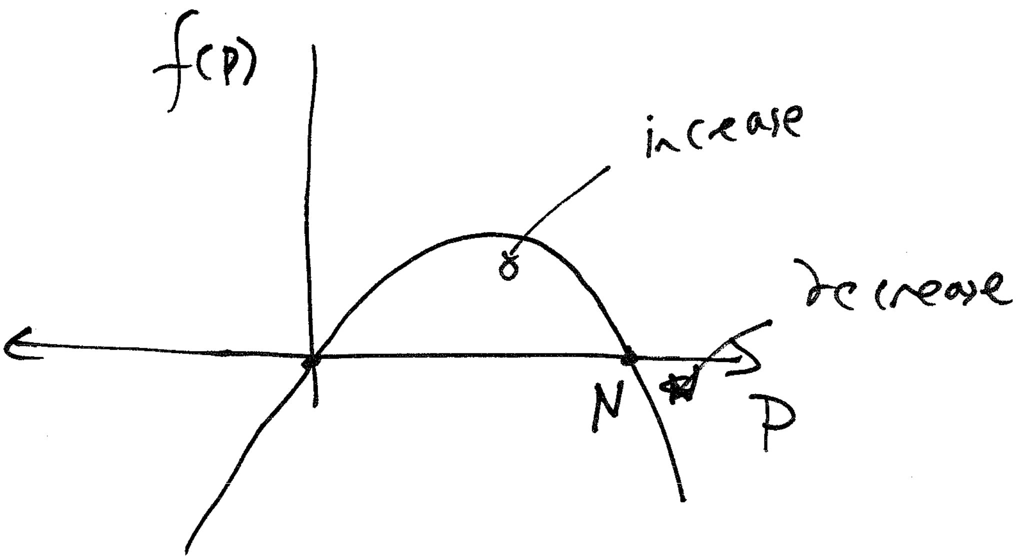
$$\frac{dP}{dt} = kP \times \text{something}$$

where 'something' ≈ 1 if P small
and 'something' < 0 if $P > N$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right), \quad k > 0$$

Predictions of the model:

$$\frac{dP}{dt} = kP(1 - P/N) = f(P)$$



$\frac{dP}{dt} > 0 \Rightarrow$ population increasing

$\frac{dP}{dt} < 0 \Rightarrow$ population decreasing

Important ideas/words from today

differential equation
ordinary differential equation
model
independent variable
dependent variable
parameter
first order differential equation
initial condition
qualitative analysis

Maths 260 Lecture 2

Topics for today

Getting started in the lab
Solutions to differential equations
Separable differential equations

Reading for this lecture

BDH Section 1.2

Suggested Exercises

BDH Section 1.2: 1, 3, 7, 15, 25

Today's handouts

Computer Laboratories for Mathematics and Statistics
An introduction to software used in the course
Lecture 2 notes

Getting started in the lab

There will be a tutorial to help you get started in the lab on

Monday, 6th March.

This week you should:

1. Make sure you know your NetAccount username and password
2. Find the lab
3. Book a computer
4. Login
5. Open 'Matlab'
6. Learn how to print

Lab Hours

Mon-Thur: 9am - 8pm, Friday: 9am - 5pm.

Section 1.2 Analytic Technique: Separation of Variables

Standard form for a first order DE is

$$\frac{dy}{dt} = f(t, y)$$

A solution of the DE is a function of the independent variable that, when substituted for the dependent variable in the DE, satisfies the DE for all values of the independent variable.

i.e. $\phi(t)$ is a solution if $\dfrac{d\phi}{dt} = f(t, \phi)$ for all t.

Example

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}$$

Which of the following functions is a solution?

1. $y_1(t) = t + 1$ → soln
2. $y_2(t) = 1 + 2t$ → No
3. $y_3(t) = 1$ → soln

e.g. 1 $\frac{dy_1}{dt} = 1$, & $\frac{y_1^2 - 1}{t^2+2t} = \frac{(t+1)^2 - 1}{t^2+2t} = \frac{t^2+2t}{t^2+2t} = 1$

i.e. $\frac{dy_1}{dt} = \frac{y_1^2 - 1}{t^2+2t}$

Initial Value Problems

An **initial condition** tells us the value of a solution to a DE at a particular value of the independent variable.

A DE with an initial condition is called an **Initial Value Problem (IVP)**.

e.g. $$\frac{dy}{dt} = te^{t^2} + 3, \qquad y(0) = -1$$

$\Rightarrow$ soln $y = \int (te^{t^2} + 3)\,dt + c$

The function $y(t) = \frac{1}{2}e^{t^2} + 3t + c$ is a solution to the differential equation for all values of c, but only the choice $c = -3/2$ satisfies $y(0) = -1$.

The function $y(t) = \frac{1}{2}e^{t^2} + 3t + c$ is the **general solution** to the DE because we can use it to solve any IVP for this DE by correctly choosing c.

Separable Equations

It is usually not possible to find analytic solutions to a DE, but there are a few special cases when we can calculate explicit solutions. Separable equations are one such case.

A DE is called **separable** if

$$\frac{dy}{dt} = f(t, y) = g(t)h(y)$$

for some functions g and h.

Examples:

$$f(t,y) = \begin{cases} (t+2)y \\ \dfrac{y}{t+3} \end{cases}$$

Special cases:

$\frac{dy}{dt} = g(t) \Rightarrow y = \int g(t)\,dt + c$

$\frac{dy}{dt} = h(y)$

but $f(t,y) = ty + t^2y^2$ is not separable

If we can do these integrals we can get an expression for $y(t)$, the solution to the DE.

Solving Separable Equations

A separable DE can be written

$$\frac{dy}{dt} = g(t)h(y)$$

for some functions f and g.

If $h(y) \neq 0$, divide by $h(y)$:

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t)$$

Since y is a function of t, we get

$$\frac{1}{h(y(t))}\frac{dy}{dt} = g(t)$$

Integrate wrt t:

$$\int \frac{1}{h(y(t))}\frac{dy}{dt}dt = \int g(t)dt$$

By the chain rule, $dy = \frac{dy}{dt}dt$, so

$$\int \frac{1}{h(y)}dy = \int g(t)dt$$

Examples

(a) $\frac{dy}{dt} = t^3 y$ $= f(t, y) = h(y) \cancel{\$} g(t)$

$h(y) = y,$ $\cancel{\&}$ $g(t) = t^3$

$$\Rightarrow \int \frac{1}{h(y)}\, dy = \int g(t)\, dt$$

$$\Rightarrow \int \frac{1}{y}\, dy = \int t^3\, dt$$

(could simply $\frac{dy}{dt} = t^3 y \Rightarrow \int \frac{dy}{y} = \int t^3 dt$)

$$\Rightarrow \ln|y| = \frac{t^4}{4} + c$$

(this is not equation for y)

can simplify

$$|y| = \cancel{A} e^{t^4/4} e^{c} = A e^{t^4/4}$$

Note that $y(t) = 0$ is also a solution to this DE but it is not found by this method ("missing solution").

(b) $\frac{dy}{dt} = \frac{t}{1+y^2}$ $\Rightarrow \int (1+y^2)\,dy = \int t\,dt$

$\Rightarrow y + y^3/3 = t^2/2 + c$

(c) $\frac{dy}{dt} = -\frac{t}{y} \Rightarrow \int y\,dy = \int -t\,dt$

$\Rightarrow y^2/2 = -t^2/2 + c$

i.e. $\Rightarrow y^2/2 + t^2/2 = c$

(equation for a circle

(d) $\frac{dy}{dt} = y\sin(t) + yt$

$\int \frac{dy}{y} = \int (\sin t + t)\,dt$

$\Rightarrow \ln|y| = -\cos t + t^2/2 + c$

$(\; |y| = A e^{-\cos t + t^2/2} \;)$

also have missing soln $y = 0$

(c) $\frac{dy}{dt} = e^{t^2}$

$$\Rightarrow \int dy = \int e^{t^2}\, dt$$

$$\Rightarrow \cancel{y = \frac{e^{t^2}}{2t}} + c$$

can not be "solved".

Important ideas/words from today

solution to a DE
initial value problem
general solution
separable equation
autonomous equation
missing solution

Maths 260 Lecture 3

Topic for today

More on separable equations

Reading for this lecture

BDH Section 1.2 (again)

Suggested Exercises

BDH Section 1.2: 35, 40

Reading for next lecture

BDH Section 1.3

Today's handout

Lecture 3 notes

(recall separable $\frac{dy}{dt} = h(y)g(t)$ $\Downarrow$

solve $\displaystyle\int \frac{dy}{h(y)} = \int g(t)\,dt + c$

Example: Model of Student Loan

A student has a student loan of \$20,000 when she completes her degree.

For the next two years the student makes no repayments and the loan accumulates interest at 8% per year. Thereafter, the student pays off \$3,600 per year and the interest rate remains at 8%.

When will she finish paying off the loan?

Setting up the model:

Assumptions

interest rate constant

continuously compounding

Variables

L=size of loan (dependent variable) in $$

t=time (independent variable) in years

$\frac{dL}{dt}$ proportional to L (loan) & r (interest rate) ($t < 2$ ⇒ no payment)

ie. $\boxed{\frac{dL}{dt} = rL = 0.08L}$, $t \leq 2$

(first two years no repayment)

For $t > 2$

$$\boxed{\frac{dL}{dt} = \underbrace{rL}_{\text{interest}} - \underbrace{3600}_{\text{payment per year}}} \quad t > 2$$

Method of solution

We can regard this model as 2 DEs:

$$\frac{dL}{dt} = 0.08L, \qquad 0 \le t < 2, \quad (1)$$

$$\frac{dL}{dt} = 0.08L - 3600, \qquad 2 \le t. \quad (2)$$

Note that both equations are separable.

Case (1): $0 \le t < 2$

$$\frac{dL}{dt} = 0.08L \Rightarrow \int \frac{dL}{L} = \int 0.08t + c$$

$$\Rightarrow \ln|L| = 0.08t + c$$

take exponential of both sides

$$e^{\ln|L|} = e^{0.08t + c}$$

$$\Rightarrow |L| = e^{0.08t} e^{c}$$

$$\Rightarrow L = \pm e^{c} e^{0.08t}$$

$$= A e^{0.08t}$$

We know $L(0)$ = original loan = 20000

$\Rightarrow A = 20000$

$\Rightarrow L = 20000e^{0.08t}$

Case (2): $2 \leq t$

$$\frac{dL}{dt} = 0.08L - 3600$$

$$\int \frac{dL}{0.08L - 3600} = \int dt$$

$$\frac{1}{0.08}\ln|0.08L - 3600| = t + c$$

$$\ln|0.08L - 3600| = 0.08t + 0.08c$$

$$|0.08L - 3600| = e^{0.08t}e^{0.08c}$$

$$0.08L - 3600 = Ae^{0.08t}$$

$$L = \frac{3600}{0.08} + \frac{A}{0.08}e^{0.08t}$$

$$= \frac{3600}{0.08} + Be^{0.08t}$$

Soln $\quad L = 20000\, e^{0.08t}, \quad t < 2$

$$L = \frac{3600}{0.08} + B e^{0.08(t-2)}, \quad t \geq 2$$

L has to be equal at ~~t≥2~~ $t = 2$

$$\Rightarrow \quad 20000\, e^{0.08.2} = \frac{3600}{0.08} + B\, \cancel{e^{0.08}}$$

$$\left(B e^{0.08t - 0.16} = B e^{0.08t} e^{-0.16} = B' e^{0.08t} \right)$$

$$B = 20000\, e^{0.16} - \frac{3600}{0.08} = -18346$$

$$L = \begin{cases} 20000 e^{0.08t}, & t < 2 \\ \dfrac{3600}{0.08} - \overset{21530}{\cancel{18346}}\, e^{0.08(t-2)}, & t \geq 2 \end{cases}$$

When is loan paid off

$$\frac{3600}{0.08} - 21530\, e^{0.08(t-2)} = 0$$

$$e^{0.08(t-2)} = \frac{3600/0.08}{21530}$$

$$0.08(t-2) = \ln\left(\frac{3600/0.08}{21530}\right)$$

$$\Rightarrow \cancel{t-2} \quad t = 2 + \frac{1}{0.08}\ln\left(\frac{3600/0.08}{21530}\right)$$

$$= 11.2 \text{ years.}$$

Using the software provided with the textbook

The textbook contains a CD of software useful for investigating DEs.

The programs on the CD are described in the Preface to the textbook.

These programs are different from the Matlab routines you will mainly use in the computer laboratory to do assignment questions.

You can use the software to investigate the behaviour of solutions to DEs you see in lectures, in the textbook, and in assignments.

Maths 260 Lecture 4

Topics for today

Slope fields

Euler's method

Reading for this lecture

BDH Sections 1.3, 1.4

Suggested Exercises

BDH Section 1.3: 11, 13, 15, Section 1.4: 7

Reading for next lecture

BDH Section 1.4 (again)

Today's handout

Lecture 4 notes

Assignment 1 Question sheet

Tutorial 1 Question sheet

Note: Monday's class will be a tutorial in the First Floor Teaching Laboratory, looking at the use of MATLAB for Assignment 1.

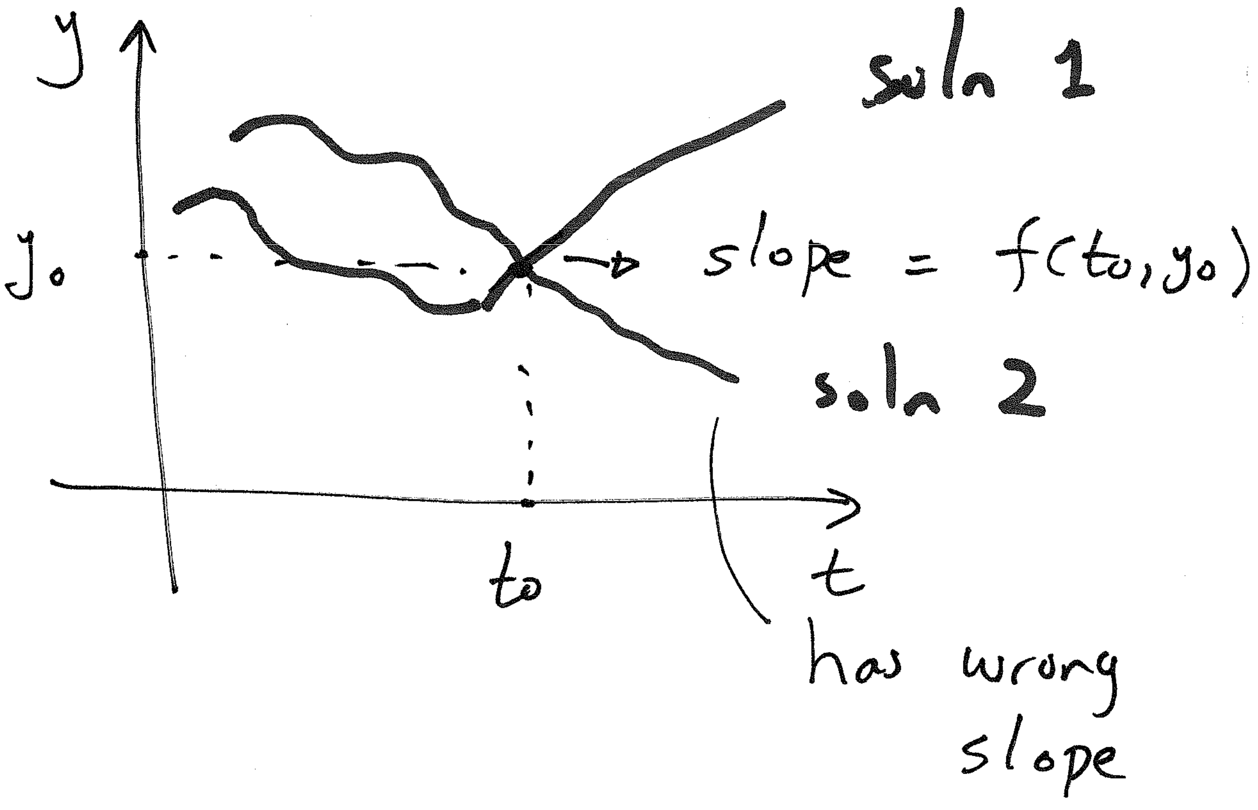
(Basement teaching Laboratory)

Section 1.3 Qualitative technique: Slope Fields

Slope fields provide a geometric technique for visualising the graph of a solution to

$$\frac{dy}{dt} = f(t, y)$$

without needing to first find a formula for $y(t)$.



Assume $y(t)$ is a solution to

$$\frac{dy}{dt} = f(t, y).$$

Then at $t = t_1$, $y(t_1) = y_1$, and

$$\frac{dy}{dt} = f(t_1, y_1)$$

i.e., slope of the graph of y at t_1 is $f(t_1, y_1)$.

Similar results for all other values of t, i.e., the slope of the graph of a solution $y(t)$ at $t = \bar{t}$ with $y(\bar{t}) = \bar{y}$ is given by $f(\bar{t}, \bar{y})$.

We use this result to draw a <u>slope field</u> which helps us sketch solutions to the DE.

<u>To draw a slope field:</u>

1. For selected points in the $t - y$ plane (say at all points on an evenly spaced grid) calculate $f(t, y)$.

2. For each point $(\bar{t}, \bar{y})$ selected in (1), draw a short line segment of slope $f(\bar{t}, \bar{y})$ centered at $(\bar{t}, \bar{y})$.

The resulting picture is called a slope field for the DE.

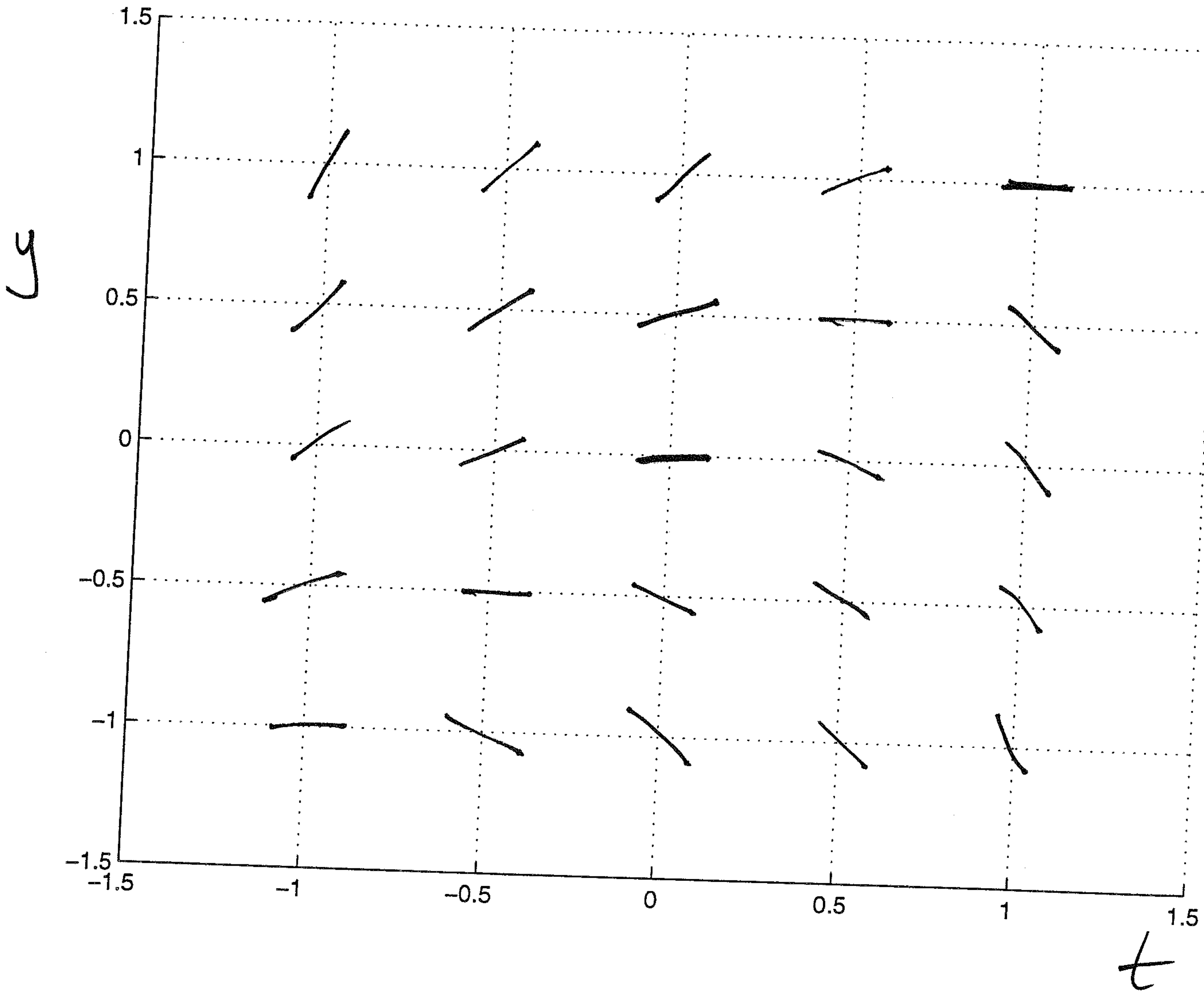
e.g.

/	$f(t,y) = 1$
\	$f(t,y) = -1$
|	$f(t,y) = -3$

Examples

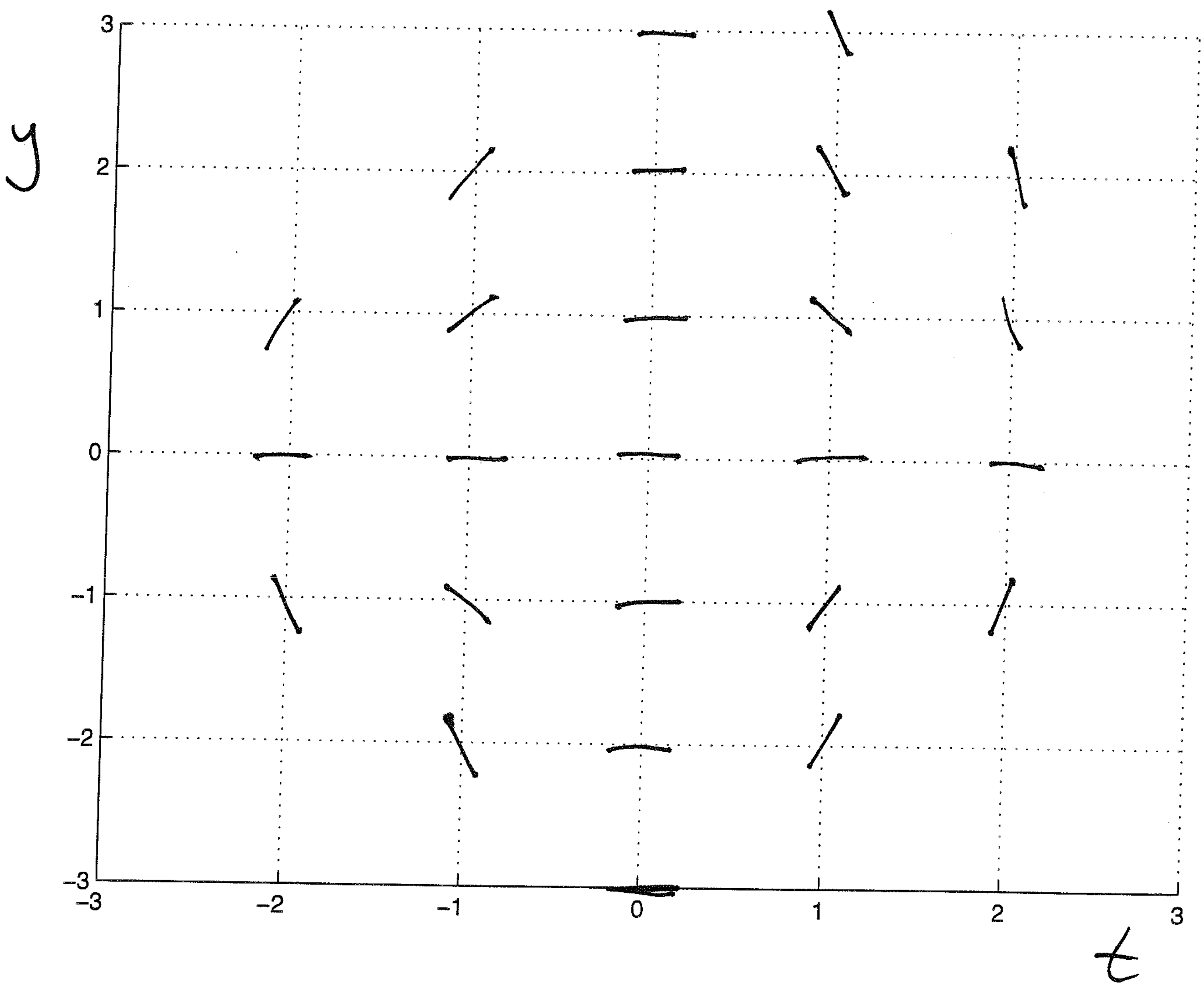
1. Use the following grid to draw the slope field for the DE

$$\frac{dy}{dt} = y - t$$



2. Use the following grid to draw the slope field for the DE

$$\frac{dy}{dt} = -yt$$



To sketch a solution using the slope field

To sketch a solution to an IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

1. Sketch the slope field as above. (or with computer)

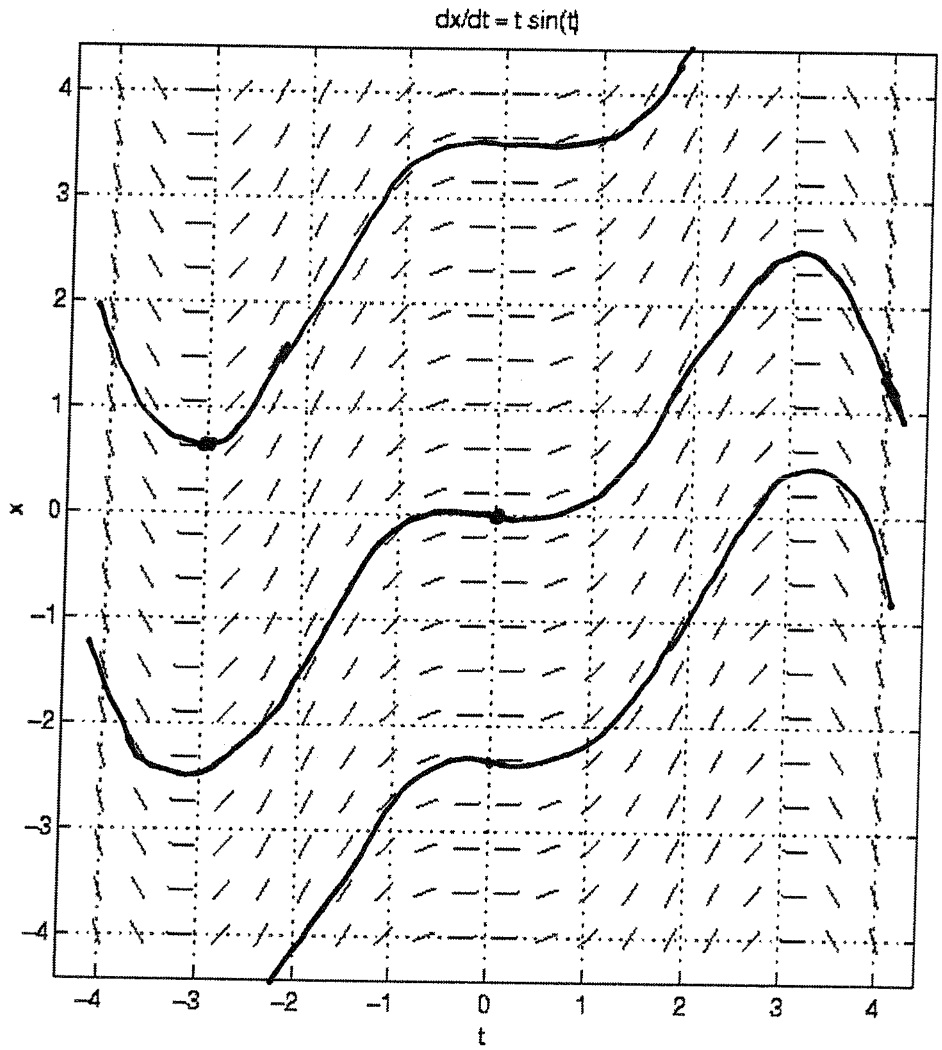
2. Starting at the point (t_0, y_0) draw a curve that follows the direction field.

Examples

1. The following picture shows the slope field for the DE

$$\frac{dy}{dt} = t \sin t$$

Draw solutions to this DE satisfying initial conditions (a) $y(1) = 0$,(b) $y(0) = -1$.

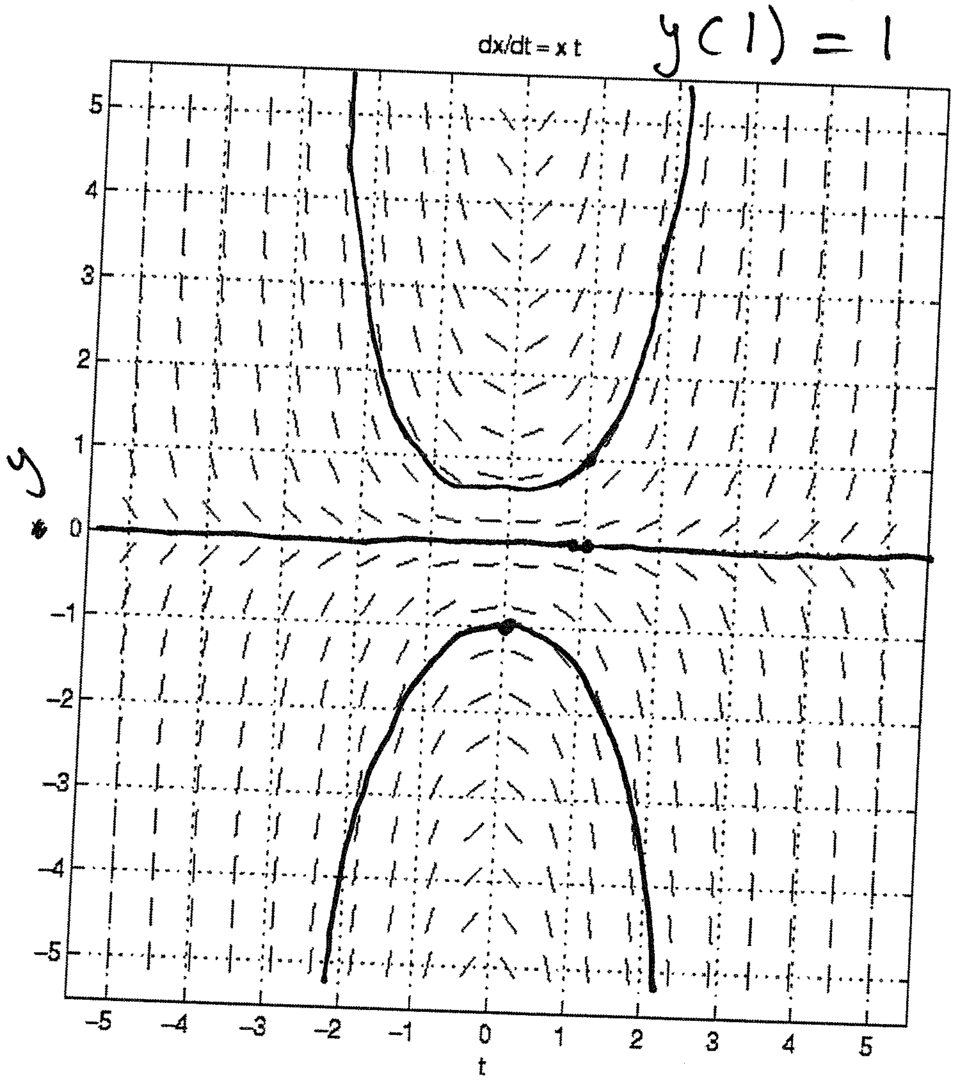


2. The following picture shows the slope field for the DE

$$\frac{dy}{dt} = yt$$

Draw solutions to this DE satisfying initial conditions (a) $y(1) = 0$ and (b) $y(0) = -1$.

$y(1) = 1$



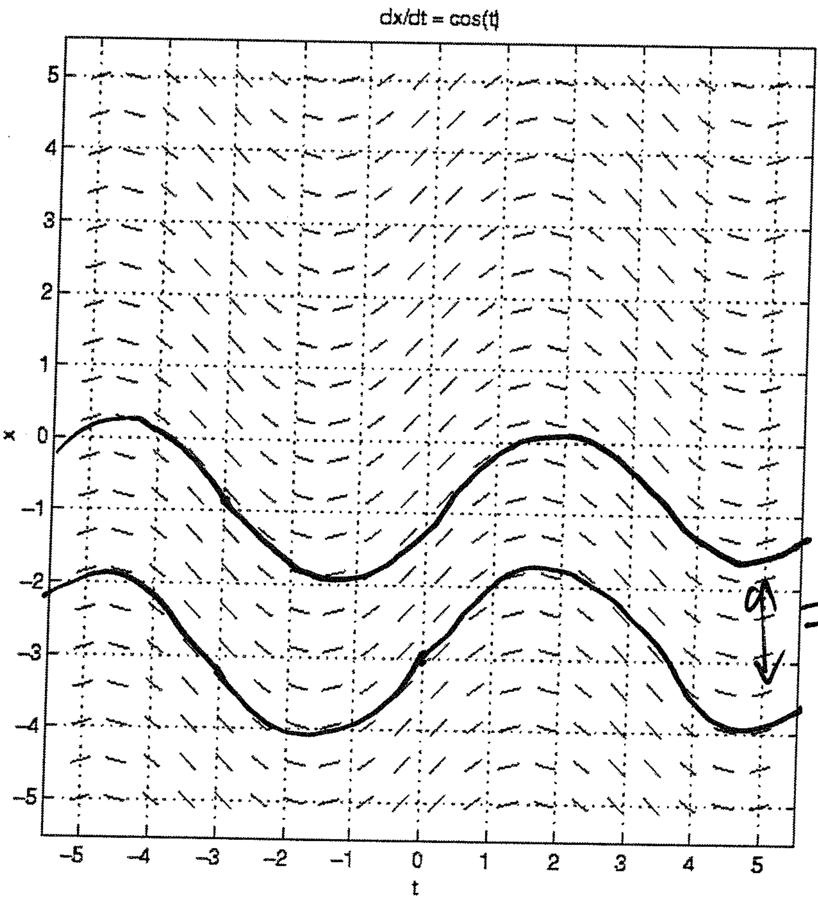
Two special cases

1. For differential equations of the form

$$\frac{dy}{dt} = f(t)$$

all slope marks on each line of fixed t in the slope field are parallel.

Example: $\dfrac{dy}{dt} = cos(t).$



$= +C$ constant from integration

Note that if we have graph of one solution, we can get graphs of other solutions by translating graph vertically.

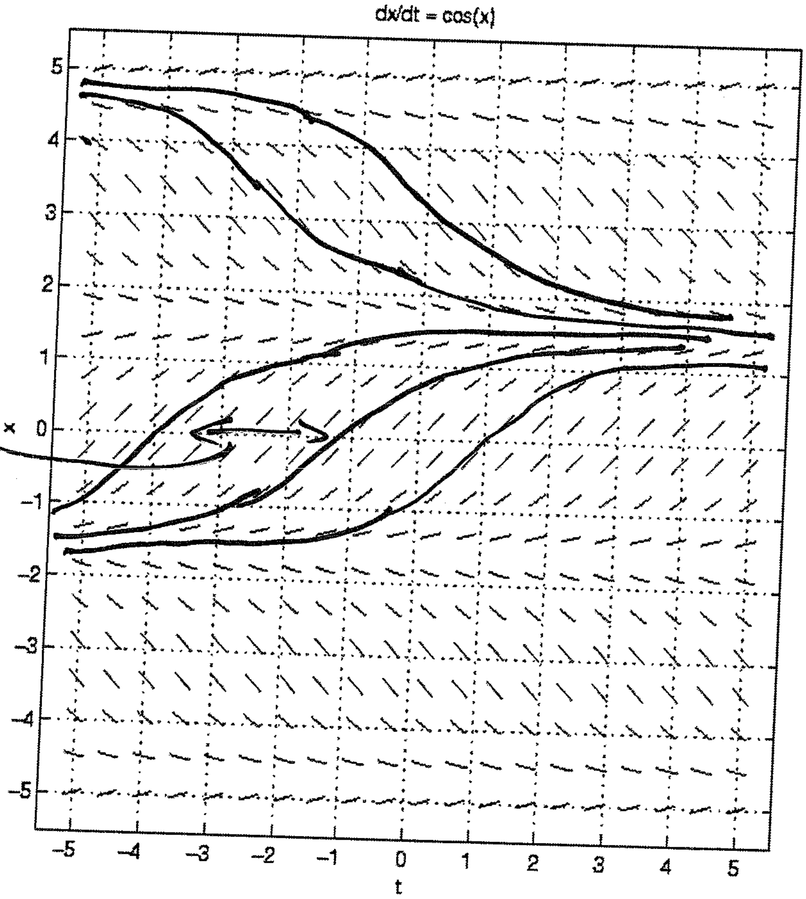
2. For differential equations of the form

$$\frac{dy}{dt} = f(y)$$

all slope marks on each line of fixed x in the slope are parallel.

Example: $\dfrac{dy}{dt} = cos(y)$.

shift



If we have the graph of one solution we can get graphs of other solutions by translating graph horizontally.

Section 1.4 Euler's method

We can obtain numbers and graphs that approximate solutions to initial value problems using a class of techniques called numerical methods. Euler's method is the simplest numerical method, and is closely related to the slope field.

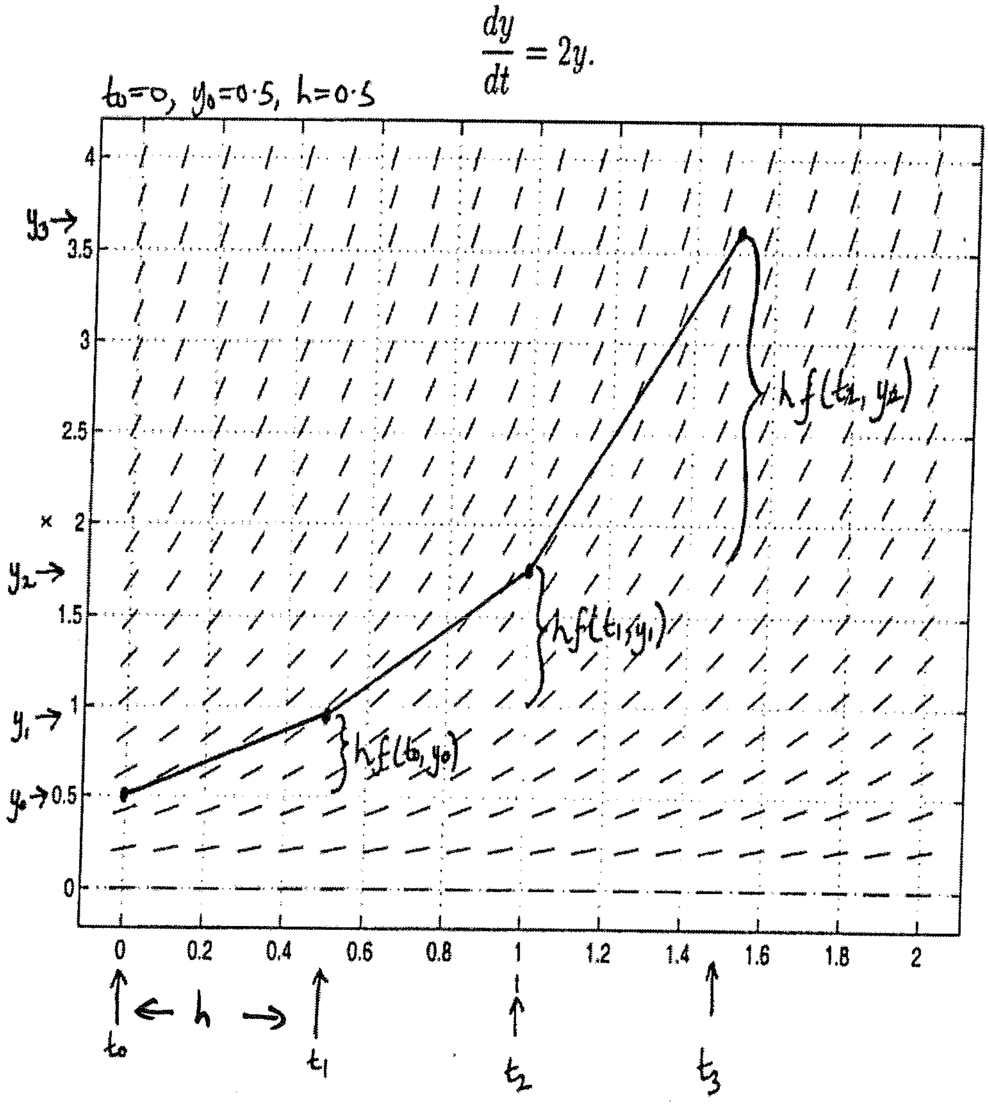
Main idea

For the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

start at (t_0, y_0) and take small steps, with the direction of each step being the direction of the slope field at the start of that step.

The following picture illustrates the relationship between the slope field and the numerical solution obtained from Euler's method for the DE $dy/dt = 2y$.



More formally, given (t_0, y_0) and a stepsize, h, we want to calculate an approximation to $y(t_1)$, $y(t_2)$, $y(t_3)$, etc where

$$\begin{aligned} t_1 &= t_0 + h, \\ t_2 &= t_1 + h = t_0 + 2h, \end{aligned}$$

and so on.

Slope of field at (t_0, y_0) is $f(t_0, y_0)$ so

$$\begin{aligned} y(t_1) &\approx y_1 = y_0 + hf(t_0, y_0), \\ y(t_2) &\approx y_2 = y_1 + hf(t_1, y_1), \\ &\vdots \\ y(t_{k+1}) &\approx y_{k+1} = y_k + hf(t_k, y_k) \end{aligned}$$

for $k = 0, 1, \ldots .n$.

Example 1: Use Euler's method to approximate the solution of the IVP

$$\frac{dy}{dt} = \sqrt{t^2 + y^2}, \quad y(0) = 0.75$$

at $t = 0.25, \quad 0.5, \quad 0.75, \quad 1.$

Solution :

$$t_0 = 0, \; t_1 = 0.25, \; t_2 = 0.5 \text{ etc}$$

$\cancel{y(0.25) \approx y_1}$

$$y_0 = 0.75, \; h = 0.25, \; f = \sqrt{t^2 - y^2}$$

$$y(0.25) \simeq y_1 = y_0 + h\, f(t_0, y_0)$$

$$= y_0 + h \sqrt{t_0^2 + y_0^2}$$

$$= 0.75 + 0.25 \sqrt{0^2 + (0.75)^2}$$

$$= 0.9375$$

$$y(0.5) \simeq y_2 = y_1 + h\, f(t_1, y_1)$$

$$= 0.9375 + 0.25 \sqrt{(0.25)^2 + (0.9375)^2}$$

$$= 1.\cancel{5}1801$$

$$y(0.75) \simeq y_3 = y_2 + h\, f(t_2, y_2)$$

$$= y_2 + h \sqrt{t_2^2 + y_2^2}$$

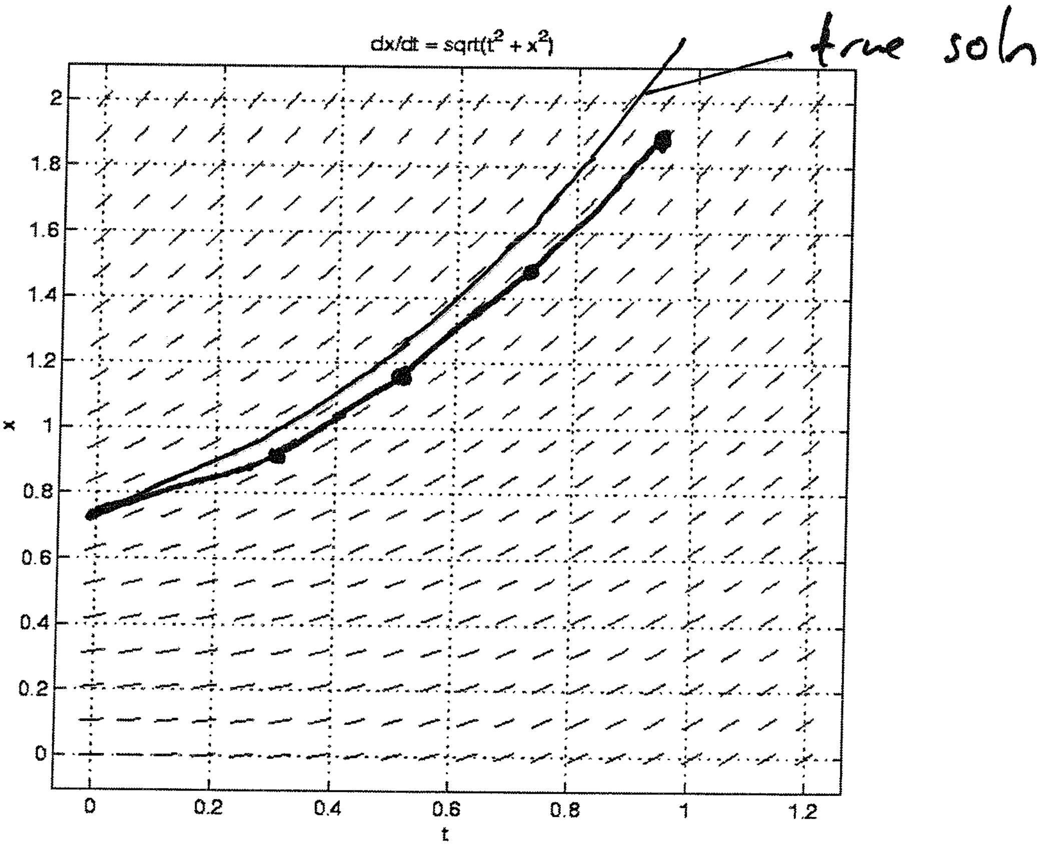
$$= 1.5006$$

$y(1.0) =$ homework

The results are best shown in the form of a table:

n	t_n	y_n	$f(t_n, y_n)$	$y_n + hf(t_n, y_n)$
0	0.0	0.75	0.75	0.9375
1	0.25	0.9375	0.9703	1.1801
2	0.50	1.1801	1.2816	1.5005
3	0.75	1.5005	1.6775	1.9198
4	1.00	1.9198		

We can compare this approximation with the solution sketched using the slope field:



Important ideas/words from today

Drawing slope fields

Sketching solutions using slope fields

Special cases of slope fields

Euler's method approximates solutions to an IVP

Euler's method is based on considering slope fields