Maths 260 Lecture 23

Topics for today

Linear systems with repeated eigenvalues

Linear systems with zero eigenvalues

Reading for this lecture

BDH Section 3.5

Suggested exercises

BDH Section 3.5; 1, 3, 5, 7, 11, 21

Reading for next lecture

BDH Section 3.7

Today's handout

Lecture 21 notes

Assignment 4 question sheet

2.8 Special Cases of Linear Systems

Linear systems with repeated eigenvalues

Example : Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{Y}$$

(fully decoupled).
Eigenvalues are 2 and 2.
Eigenvectors:

$$\lambda = 2 \quad \begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \vec{u} = 0$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{u} = 0$$

$\vec{u} =$ any vector.

e.g. $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

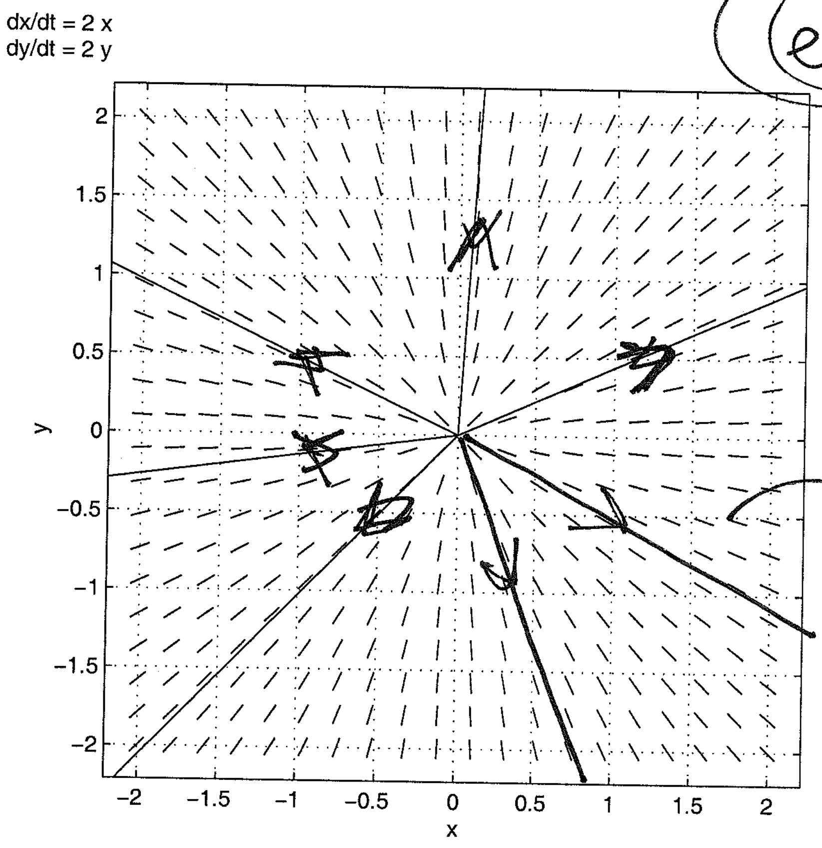
The general solution is:

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

i.e., every non-zero solution is a straight-line solution.

Phase portrait

~~e^{2t} = source~~



only have straight line solns.

This example illustrates a general case:

If matrix $\mathbf{A}$ has a repeated eigenvalue λ with two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then

$$\mathbf{Y}_1 = e^{\lambda t}\mathbf{v}_1$$

and

$$\mathbf{Y}_2 = e^{\lambda t}\mathbf{v}_2$$

are linearly independent straight line solutions.

We can construct a general solution from a linear combination of these two solutions as usual.

Furthermore, if $\mathbf{A}$ is a 2×2 matrix, then every solution except the equilibrium at the origin is a straight-line solution.

If $\lambda > 0$, then every non-zero solution tends to ∞ as $t \to \infty$ (so the origin is a source).

If $\lambda < 0$, then every solution tends to the origin as $t \to \infty$ (so the origin is a sink).

What happens if we cannot find two linearly independent eigenvectors?

Example Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 8 & -5 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues are -5 and -5.
Eigenvectors:

$$\begin{pmatrix} -5-\lambda & 0 \\ 8 & -5-\lambda \end{pmatrix} \vec{u} = 0$$

$\lambda = -5$

$$\begin{pmatrix} 0 & 0 \\ 8 & 0 \end{pmatrix} \vec{u} = 0$$

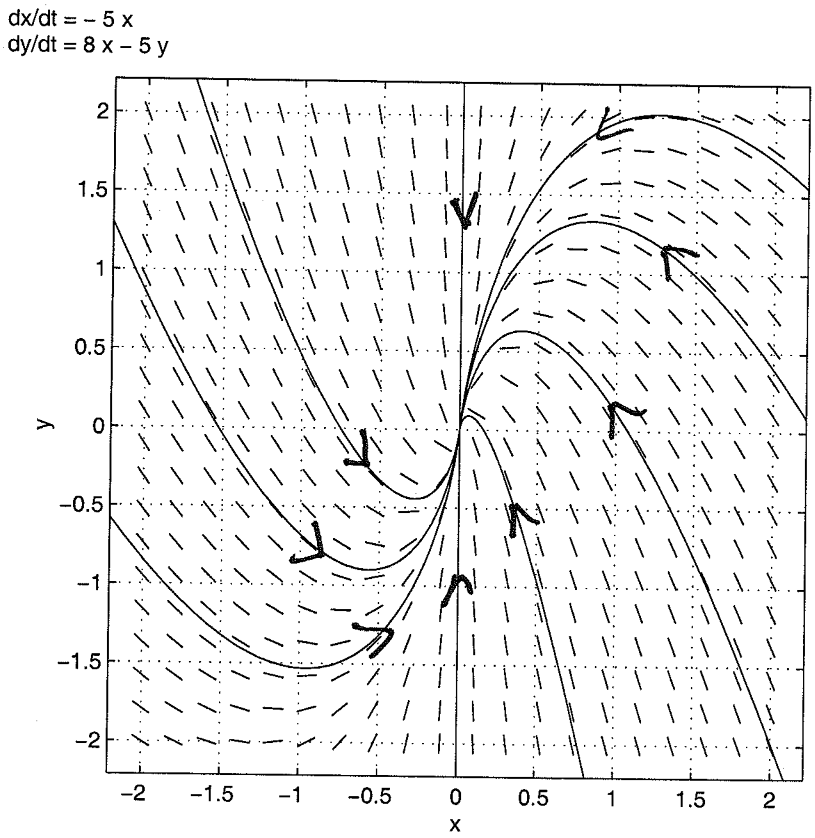
$$\vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(only one eigenvector)

$$Y = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t}$$

is a soln but not "complete soln"

Phase portrait and some solutions



See that system has only one straight line solution. We can't write the general solution as a linear combination of solutions of the form $e^{\lambda t}\mathbf{v}$ because we don't have enough such solutions.

To find a second solution, we use the following result.

Theorem: Consider the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ has a repeated eigenvalue λ with just one linearly independent eigenvector. Pick an eigenvector $\mathbf{v}_1$ corresponding to λ.

Then

$$\mathbf{Y}_1 = e^{\lambda t}\mathbf{v}_1$$

normal soln

is a straight-line solution and

$$\mathbf{Y}_2 = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$$

special second soln

is a second, linearly independent solution of the system, where $\mathbf{v}_2$ is a vector satisfying

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$

easy to solve

($\mathbf{v}_2$ is called a generalised eigenvector).

(eq for v_1 is

$$(A - \lambda I)\vec{v}_1 = 0$$

$\vec{v}_2$ is unique up to adding multiple of $\vec{v}_1$

Can use this second solution $\mathbf{Y}_2$ to construct the general solution for the previous example.

Example

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 8 & -5 \end{pmatrix} \mathbf{Y}.$$

Found already that $\mathbf{Y}_1 = e^{-5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a solution.

Look for $\mathbf{v}_2$ satisfying

$$(\mathbf{A} - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$$

$$A - \lambda I = \begin{pmatrix} -5-\lambda & 0 \\ 8 & -5-\lambda \end{pmatrix}_{\lambda=-5}$$

$$= \begin{pmatrix} 0 & 0 \\ 8 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ 8 & 0 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \quad 8\alpha + 0\cdot\beta = 1, \quad \alpha = 1/8$$

$$\underset{=0}{\beta = \text{anything}}.$$

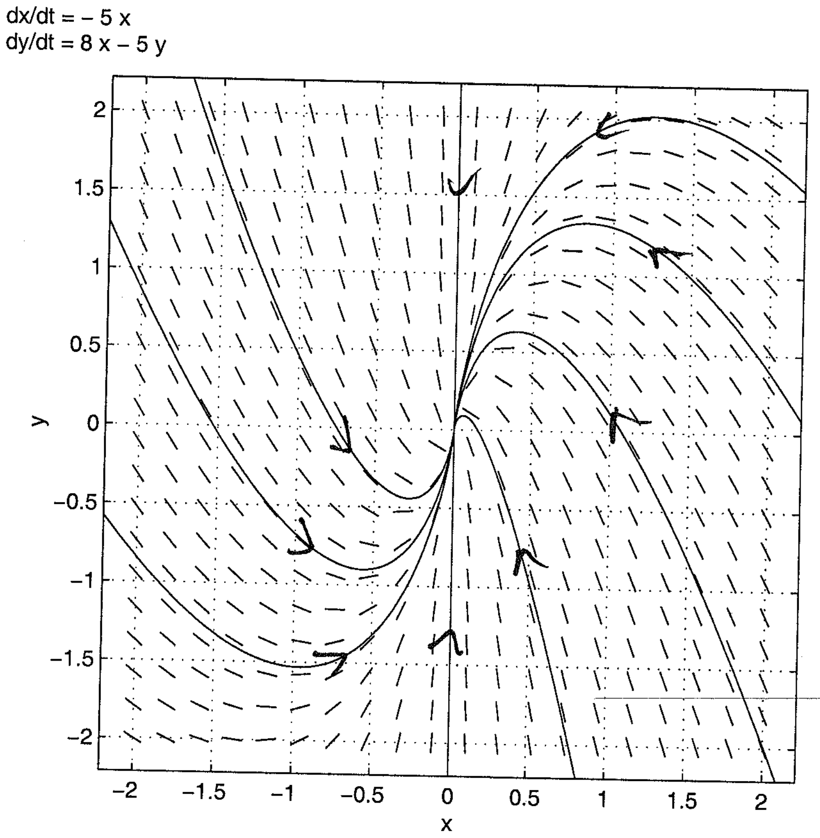
$$\vec{v}_2 = \begin{pmatrix} 1/8 \\ 0 \end{pmatrix}$$

$$Y(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 \left(t\vec{v}_1 + \vec{v}_2 \right) e^{\lambda t}$$

$$= c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t} + c_2 \left(t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/8 \\ 0 \end{pmatrix} \right) e^{-5t}$$

(eq. pt is sink
since $e^{-5t} \to 0$ faster than
$t \to \infty$)

Direction field and some solutions



We see that all solutions are tangent at the origin to the direction of the straight-line solution.

This is always the case in a 2×2 system: when there is a non-zero repeated eigenvalue with only one corresponding linearly independent eigenvector, all solution curves in the phase plane are tangent to the straight-line solution.

Important note: There is some freedom when choosing a generalised eigenvector.

For example, in previous example

$$\mathbf{v}_2 = \begin{pmatrix} \frac{1}{8} \\ y \end{pmatrix}$$

is a generalised eigenvector for any choice of y.

However, a multiple of a generalised eigenvector **is not** usually a generalised eigenvector.

For example, in previous example

$$k \begin{pmatrix} \frac{1}{8} \\ y \end{pmatrix}$$

is not a generalised eigenvector for any choice of k except $k = 1$.

Different choices of the generalised eigenvector all lead to the same general solution.

Example : Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

$$\det \begin{pmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1 = 0$$

$$\lambda = 1 \quad \text{repeated}$$

find eigenvector

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

solve for generalised eigenvector $\vec{v}_2$

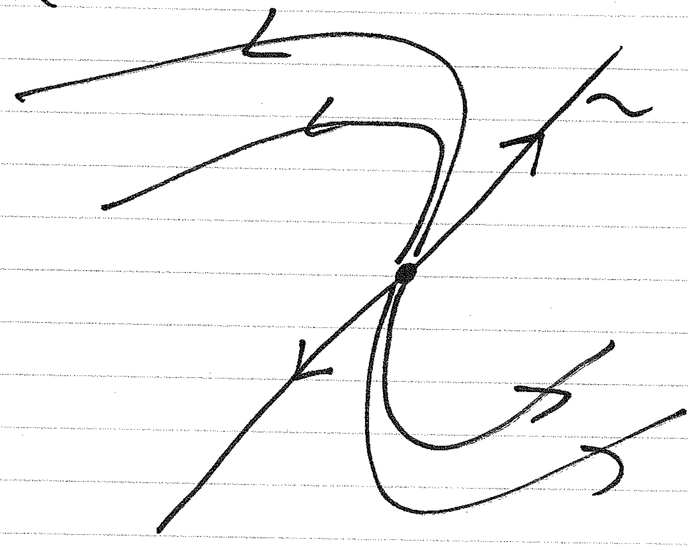
e.g. $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{v}_2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \Rightarrow \alpha - \beta = 1$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ or } \dots$$

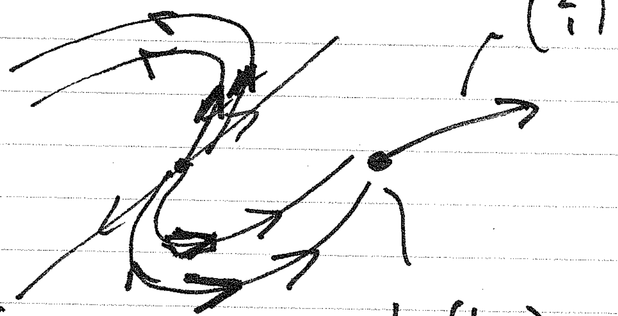
$$Y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^t$$

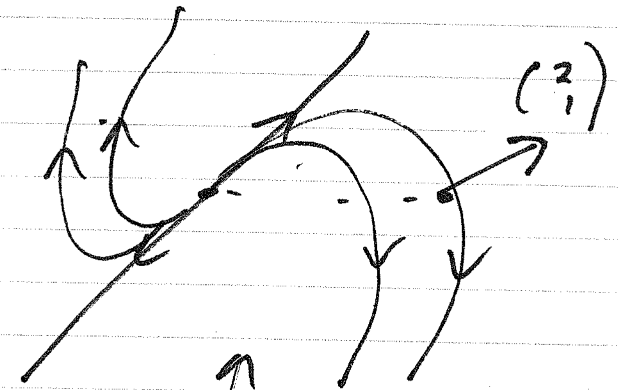
Sketch



$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Two possibilities

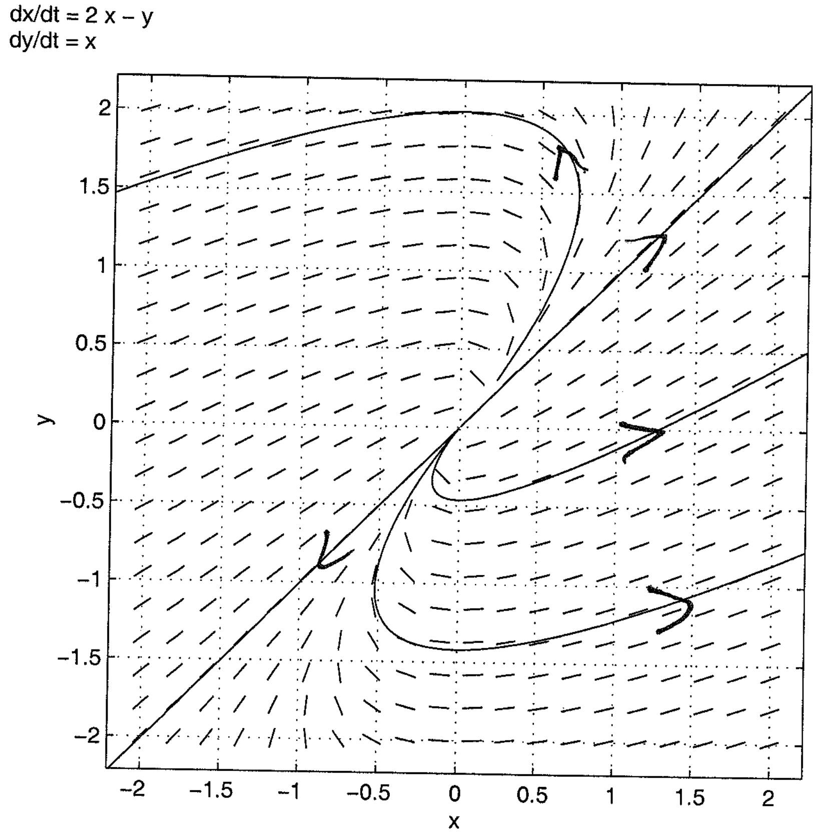




$$\frac{dY}{dt}\bigg|_{(1,0)} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

RIGHT WRONG

Direction field and some solutions



Linear systems with zero eigenvalues

Example : Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \mathbf{Y}$$

Eigenvalues are 5 and 0 with eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively.

So

$$\mathbf{Y}_1 = e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and

$$\mathbf{Y}_2 = e^{0t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

are linearly independent solutions, and the general solution is

$$\mathbf{Y}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \underbrace{c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{line is constant}} \cancel{e^{0t}}$$

If $c_1 = 0$, then

$$\mathbf{Y}(t) = c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

which is constant, so this is an equilibrium solution for all choices of c_2.

This is a general result: all points on a line of eigenvectors corresponding to a zero eigenvalue are equilibrium solutions.

If $c_1 \neq 0$ then first term in general solution tends to zero as $t \to -\infty$, i.e., solution tends to the equilibrium

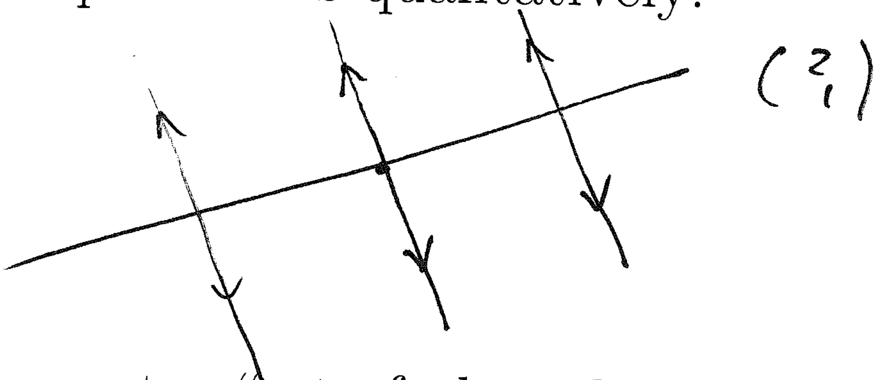
$$c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

along a line parallel to

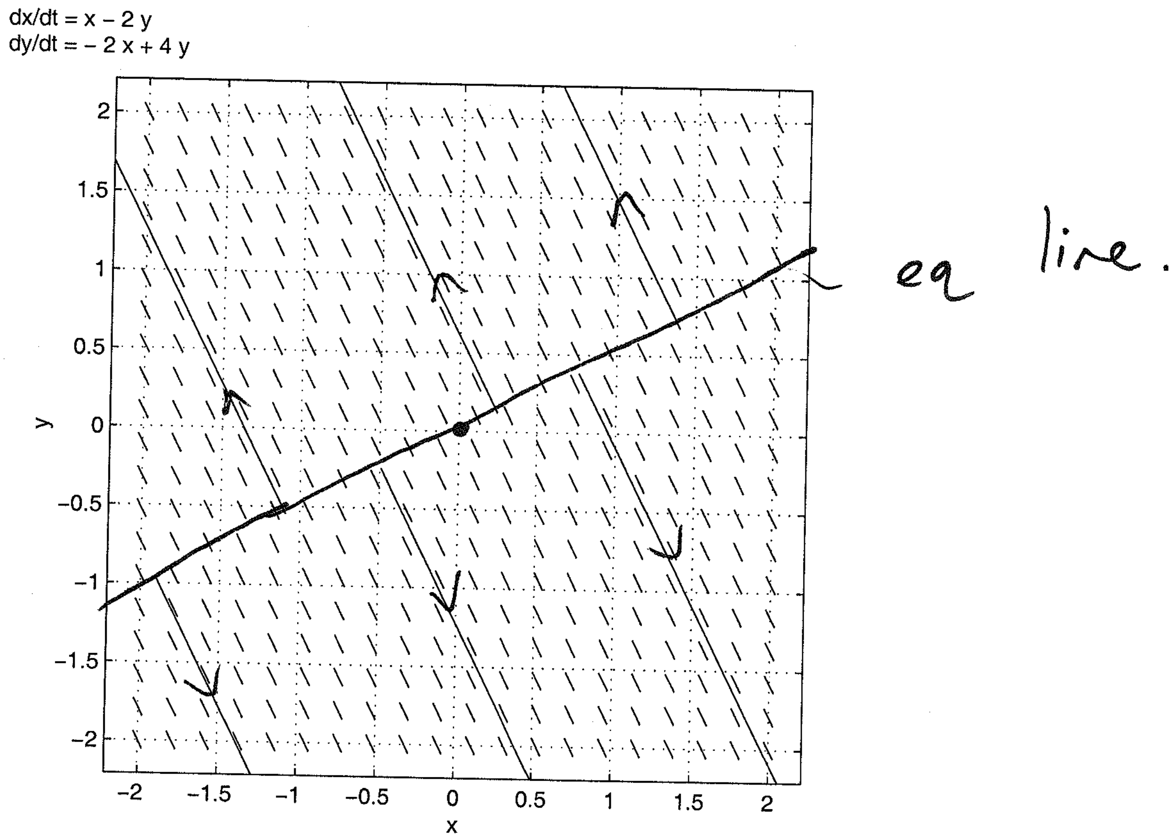
$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

as $t \to -\infty$.

Hence, phase portrait is qualitatively:



From *pplane*, get a “set of phase lines”:



Get similar behaviour in other linear systems with a zero eigenvalue, but details of the general solution and the phase portrait may vary depending on the specific example.

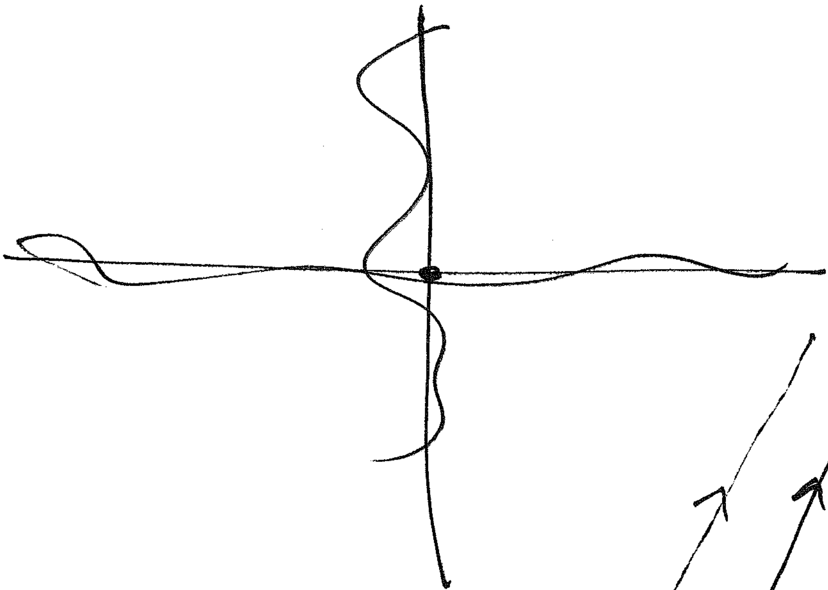
Example : Sketch the phase portrait for the system

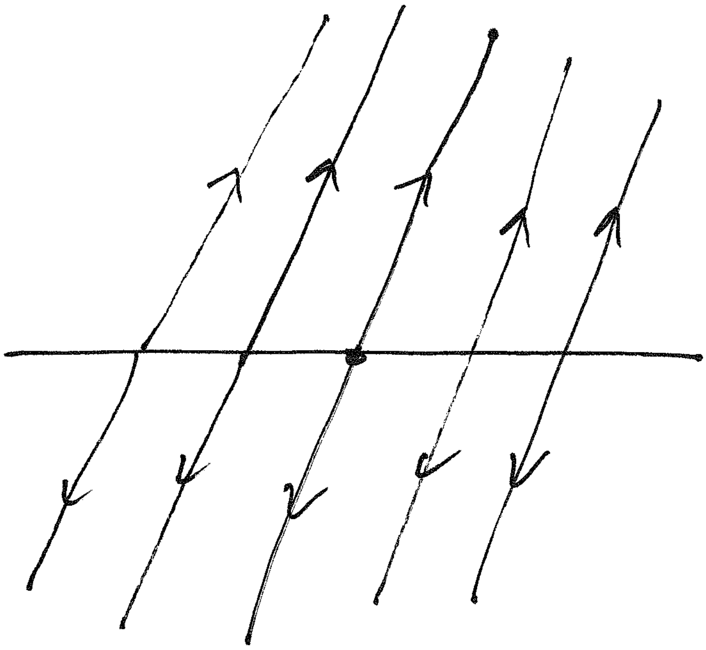
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix} \mathbf{Y}$$

Find eigenvalues & eigenvectors.

$\lambda = 0, \quad \lambda = 4$

$\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{V}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$





Maths 260 Lecture 24

Topics for today

Bifurcations in linear systems

Reading for this lecture

BDH Section 3.7

Suggested exercises

BDH Section 3.7; 2(b,c), 6(b,c)

Reading for next lecture

BDH Section 5.1

Today's handouts

Lecture 23 notes

2.9 Putting it all together - bifurcations in linear systems

Bifurcation: sudden *qualitative* change in the dynamics.
In our examples today, the following results will be useful:

For any matrix A,

$$\det(A) = \text{product of eigenvalues} = \lambda_1 \lambda_2$$

and

$$\text{trace}(A) = \text{sum of eigenvalues.} = \lambda_1 + \lambda_2$$

$$tr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

If A is a 2×2 matrix, the signs of $\det(A)$ and $\text{trace}(A)$ tell us a lot about the type of the equilibrium at the origin for the system

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$$

For example, if A is a 2×2 matrix with $\det(A) < 0$, then the origin is a saddle.

Example : Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix} \mathbf{Y}$$

where a is a parameter.

Determine the type of equilibrium at the origin for all values of a. Sketch the phase portrait for representative values of a.

$$\det\left(\begin{pmatrix} 1-\lambda & 2 \\ a & -\lambda \end{pmatrix}\right) = \lambda^2 - \lambda - 2a = 0$$

$$\Rightarrow \quad \lambda = \frac{1 \pm \sqrt{1 + \cancel{4}8a}}{2}$$

Eigenvalues of matrix

$$A = \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix}$$

are

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+8a}$$

so the type of equilibrium at the origin depends on a.

Also,

$$\det(A) = -2a$$

and

$$\mathrm{trace}(A) = 1.$$

$$\left(\lambda = \frac{1}{2}\left(1 \pm \sqrt{?}\right) \right)$$

$? < 0 \Rightarrow$ spiral (source)

$0 < ? < 1 \Rightarrow$ both $\lambda > 0 \Rightarrow$ suarce

$? > 1 \Rightarrow \lambda_1 > 0,\ \lambda_2 < 0 \Rightarrow$ saddle

We find the following qualitatively distinct cases, depending on a.

1. If $1 + 8a < 0$, eigenvalues of A are complex, say $\alpha \pm i\beta$.

$$\left(1 + 8a < 0 \Rightarrow a < -\frac{1}{8} \right)$$

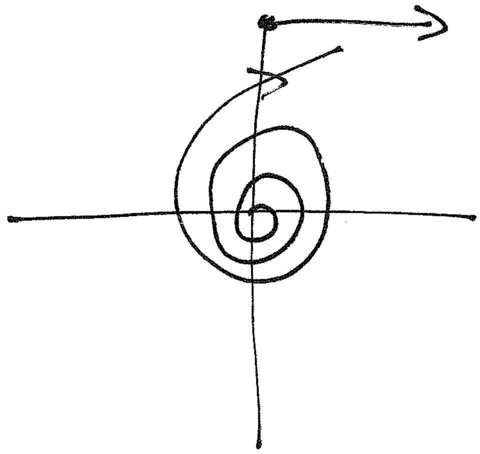
$$\cancel{\alpha \approx} \quad \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+8a}$$

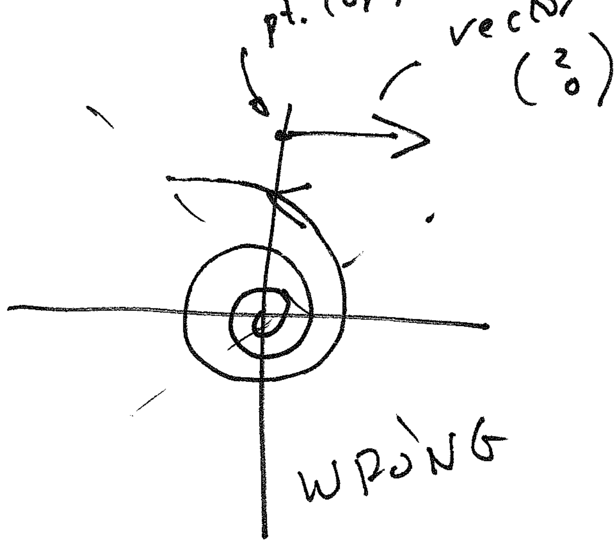
$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{2}\sqrt{-1-8a}$$

$\Downarrow$

spiral source

Two cases





Now $$\left.\frac{dY}{dt}\right|_{(0,1)} = \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad a > -\frac{1}{8}$

2. If $1 + 8a > 0$, eigenvalues of $\mathbf{A}$ are real. Subcases:

   (a) If $a > 0$, $\det(\mathbf{A}) < 0$ so there is one positive eigenvalue and one negative eigenvalue. $\Rightarrow$ saddle point

$(\det(A) = -2a)$

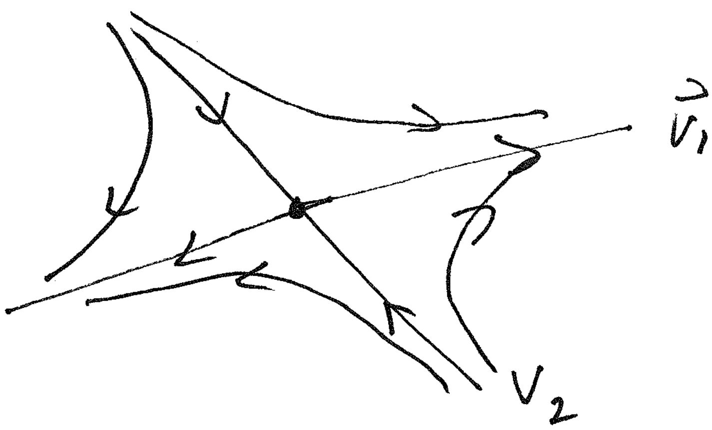
~~case a > 0~~

e.g. $a = 1$, $\dfrac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} Y$

Find eigenvalues & eigenvectors of $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$\lambda_1 = 2 \quad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



(b) If $-\frac{1}{8} < a < 0$, $\det(\mathbf{A}) > 0$, so eigenvalues are of the same sign (and real). But $\text{trace}(\mathbf{A}) > 0$ so both eigenvalues are positive. $\Rightarrow$ source

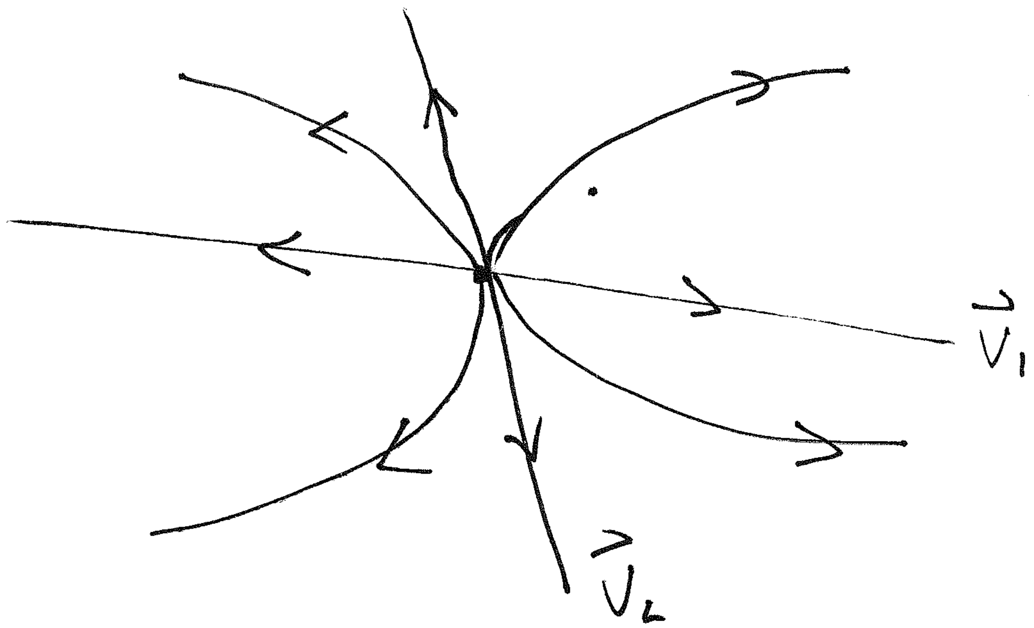
$a = -1/16$, $\frac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ -1/16 & 0 \end{pmatrix} Y$

~~eig~~

eigenvalue / eigenvectors

$\lambda_1 = 0.85$, $\vec{v}_1 = \begin{pmatrix} 0.99 \\ -0.07 \end{pmatrix}$

$\lambda_2 = 0.14$ $\vec{v}_2 = \begin{pmatrix} -0.91 \\ 0.39 \end{pmatrix}$



transition from a

3. **Transitional values of a**

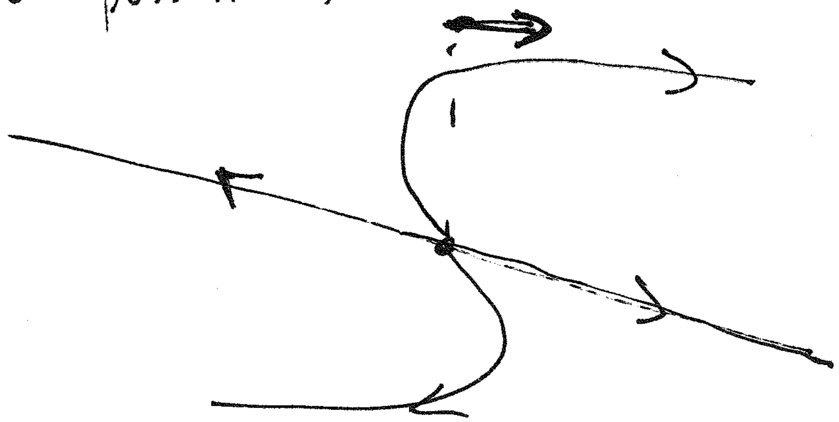
(a) $a = -\frac{1}{8}$, (a spiral source to source)

$$A = \begin{pmatrix} 1 & 2 \\ -\frac{1}{8} & 0 \end{pmatrix}$$

In this case, eigenvalues of A are $\frac{1}{2}$ (twice) with just one linearly independent eigenvector

$$\begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Two possibilities



(know $\left.\frac{dY}{dt}\right|_{(0,1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

WRONG

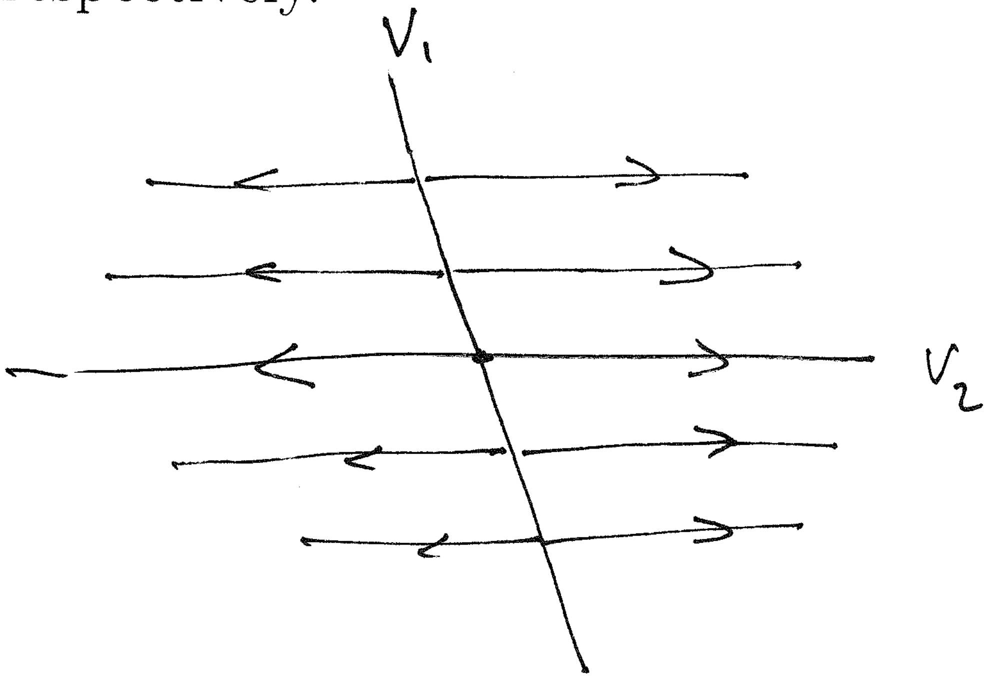
(b) $a = 0$,

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

In this case, eigenvalues of A are 0 and 1 with eigenvectors

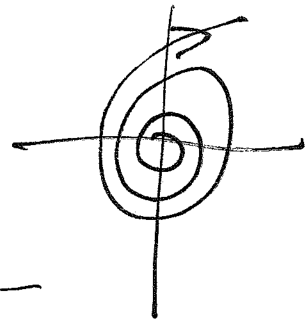
$$v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_2$$

respectively.

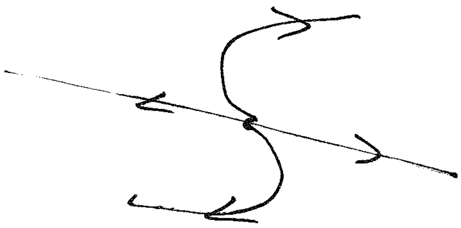


Representative phase portraits from *pplane* are given below.

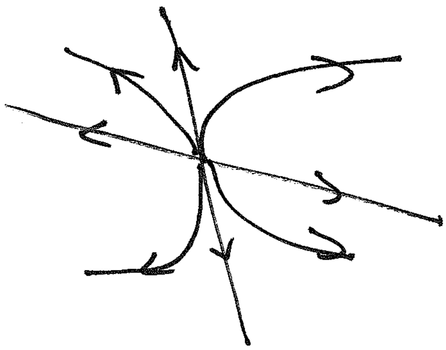
$a < -\frac{1}{8}$



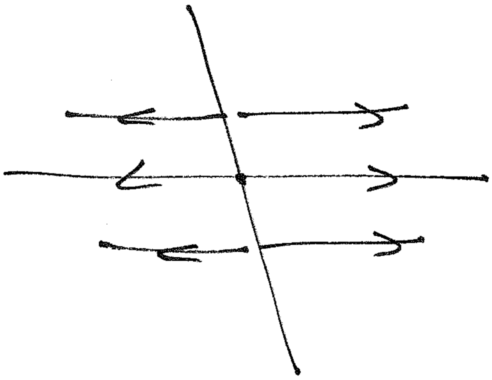
$a = -\frac{1}{8}$



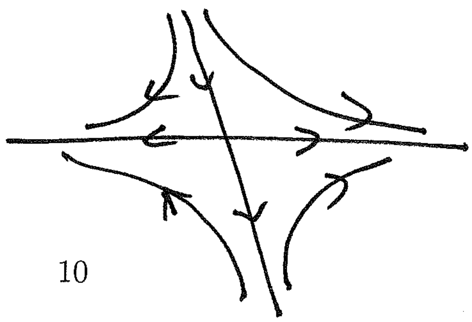
$-\frac{1}{8} < a < 0$



$a = 0$



$a > 0$



Example : Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \overbrace{\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}}^{= A} \mathbf{Y}$$

where a is a parameter.

Determine the type of equilibrium at the origin for all values of a. Sketch the phase portrait for representative values of a.

eigenvalues $= \lambda = \dfrac{a \pm \sqrt{a^2-4}}{2}$

(For $a>2$ both eigenvalues positive
since $a > \sqrt{a^2-4}$ or $\det(A) = 1 > 0$)

$a < -2$ ~~source~~ sink

$-2 < a < 0$ spiral sink

$0 < a < 2$ spiral source

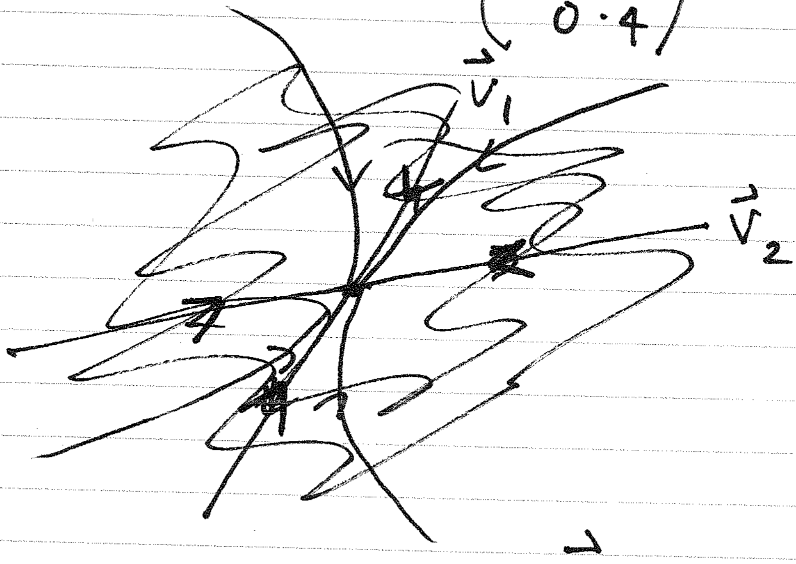
$a > 2$ ~~sin~~ source

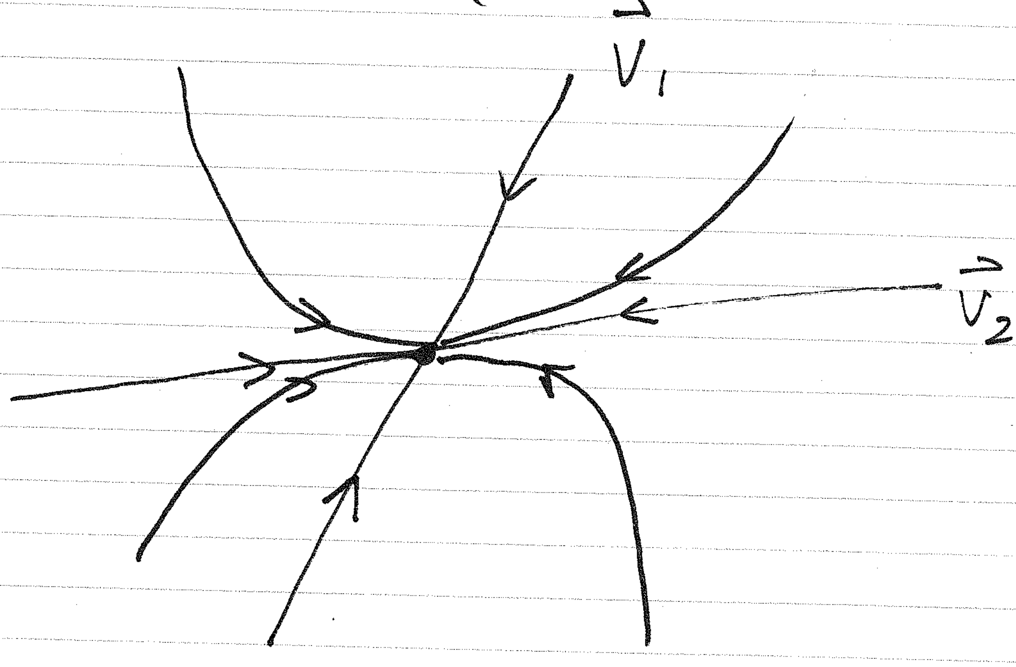
$a < -2, \quad a = -3$

$$\frac{dY}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix} Y$$

$\lambda_1 = -2.6 \quad \vec{v}_1 = \begin{pmatrix} 0.4 \\ 0.9 \end{pmatrix}$

$\lambda_2 = -0.4 \quad \vec{v}_2 = \begin{pmatrix} 0.9 \\ 0.4 \end{pmatrix}$

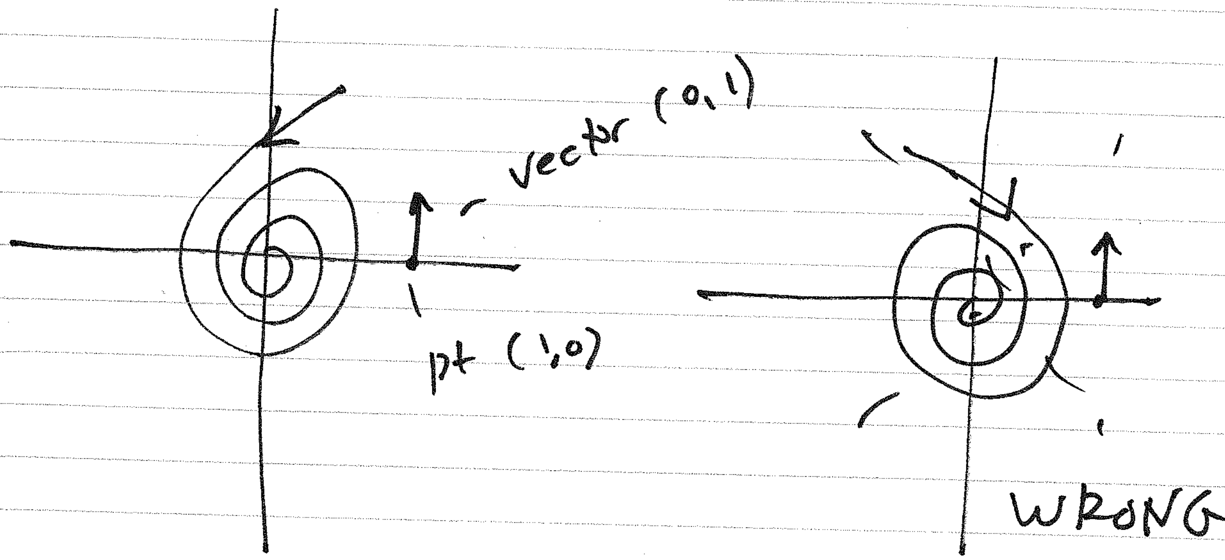




$-2 < a < 0$ (spiral sink)

(we do not need eigenvectors)

Two possibilities



$$\left.\frac{dY}{dt}\right|_{(1,0)} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From these two examples we see how the transitional cases arise as a parameter is varied:

1. a centre occurs as a spiral sink changes to a spiral source, or vice versa;
2. an improper node (i.e., two equal eigenvalues with only one linearly independent eigenvector) occurs when a spiral sink (or source) turns into a real sink (or source), or vice versa;
3. a linear system with a zero eigenvalue occurs when a saddle turns into a sink or source, or vice versa.