Maths 260 Lecture 10

Topic for today

Bifurcations

Reading for this lecture

BDH Section 1.7

Suggested exercises

BDH Section 1.7: 1, 3, 9

Reading for next lecture

BDH Section 1.7 (again)

Today's handout

Lecture 10 notes

"Tipping point"

Section 1.7: Bifurcations

Many DE models contain parameters, i.e., quantities that do not depend on the independent variable but may take on different values.

We are interested in how the behaviour of solutions (especially the long term behaviour) changes as parameters are changed. For instance,

1. What are the solutions like over a range of parameter values?

2. How good is our model if we only know the parameter value roughly?

A small change in the value of a parameter usually results in a small change in solutions.

A **bifurcation** occurs when a small change in parameter gives a qualitative change in the behaviour of solutions.

We look at autonomous equations that depend on one parameter, i.e.,

$$\frac{dy}{dt} = f_\mu(y).$$

This is a one-parameter family of DEs - we get one DE for each choice of the parameter μ.

e.g. $f_\mu(y)$ "might equal" $= \begin{cases} y^2 + 2\mu \\ y^2 + \cos\mu \\ y^2 + 3\mu y + 7 \end{cases}$

μ is fixed constant

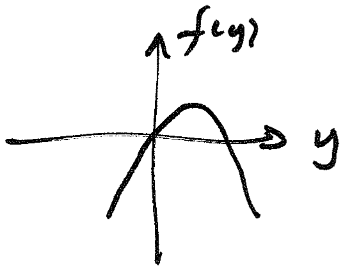
μ does not appear in L.H.S.

Example: Consider the DE

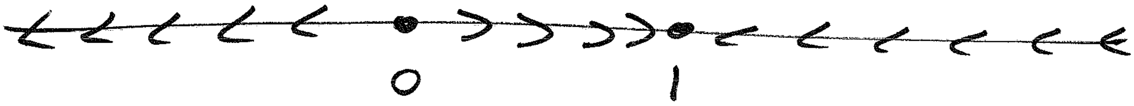
$$\frac{dy}{dt} = f_h(y) = y(1-y) - h$$

Compare the phase lines at $h = 0$ and $h = 1$:

$h = 0$

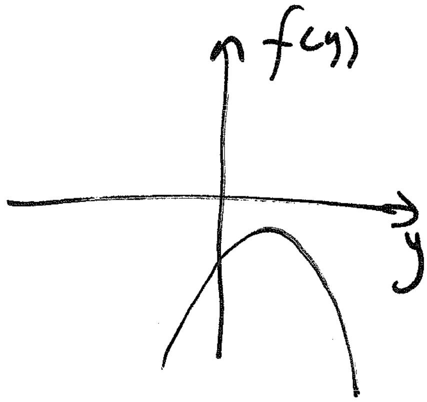


$$\frac{dy}{dt} = y(1-y)$$



$h = 1$

$$\frac{dy}{dt} = y(1-y) - 1$$



We see that there must be a bifurcation at some value of h in the interval $(0, 1)$.

We now find and classify equilibria as a function of h and hence locate the bifurcation value of h.

$$\frac{dy}{dt} = y(1-y) - h = f(y)$$

solve for equilibria

$$f(y) = 0 \Rightarrow y(1-y) - h = 0$$

$$-y^2 + y - h = 0$$

$$\Rightarrow y = \frac{-1 \pm \sqrt{1-4h}}{-2}$$

$$y = \frac{1}{2} \pm \frac{\sqrt{1-4h}}{2}$$

Work out if equilibria are sources or sinks.

Bifurcation Diagrams

A bifurcation diagram is a picture in the $\mu - y$ plane of the phase lines near a bifurcation value.

It highlights the changes that the phase lines undergo as the parameter passes through the bifurcation value.

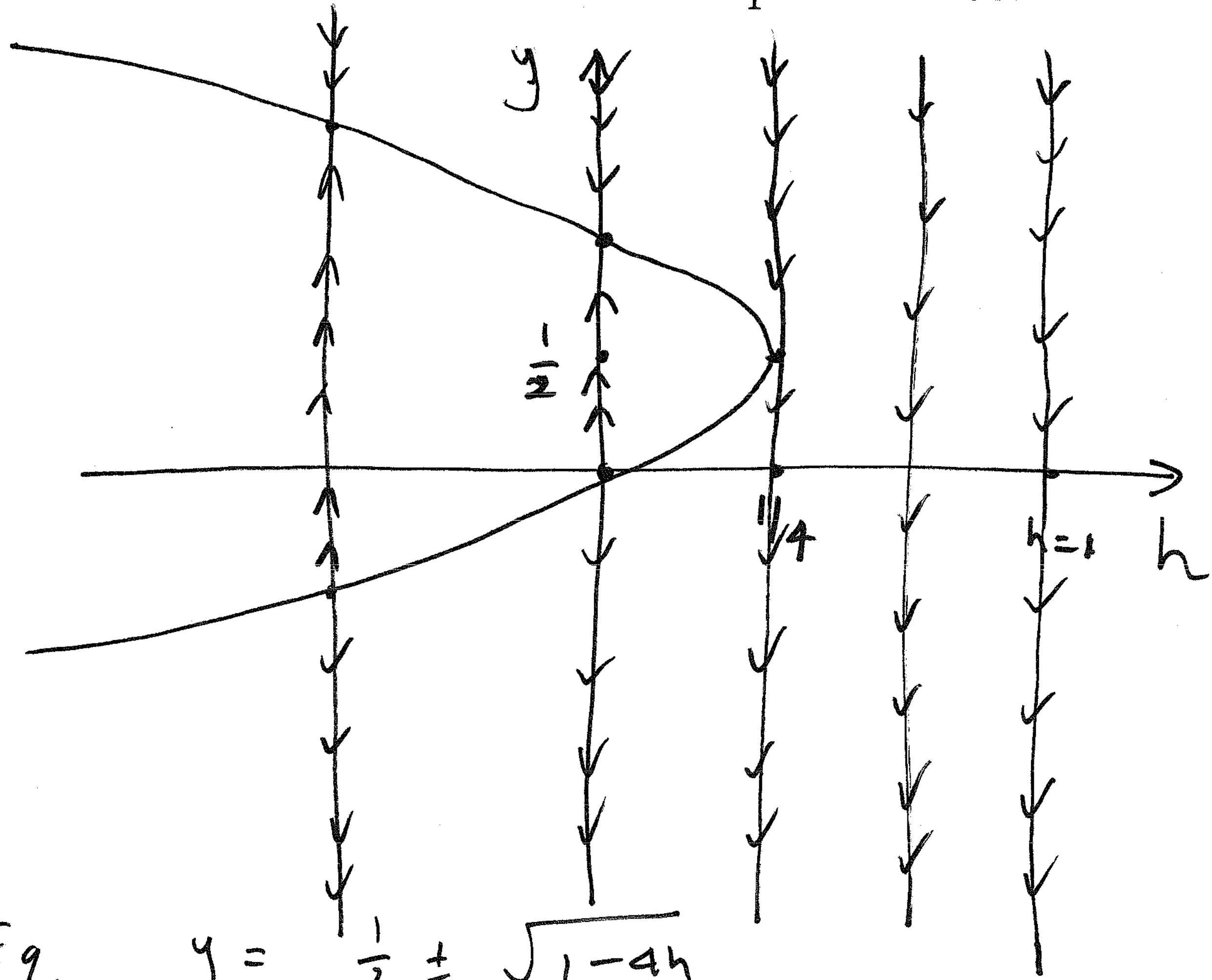
Procedure for drawing a bifurcation diagram

1. Draw μ and y axes and label them.
2. Plot curves showing position of equilibria as μ varies.
3. Sketch representative phase lines, including at least one for each of $\mu < \mu_0$, $\mu = \mu_0$, $\mu > \mu_0$ where μ_0 is a bifurcation value.
4. Label any significant values of μ and y, including bifurcation values.

Example: Draw the bifurcation diagram for the one-parameter family

$$\frac{dy}{dt} = y(1-y) - h,$$

where h is the bifurcation parameter.



Eq $y = \frac{1}{2} \pm \frac{\sqrt{1-4h}}{2}$

equilibrium pt. at $y = \frac{1}{2} + \frac{\sqrt{1-4h}}{2}$ sink

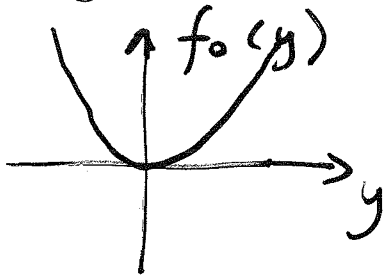
" " " $y = \frac{1}{2} - \frac{\sqrt{1-4h}}{2}$ source

Example: For the family of equations

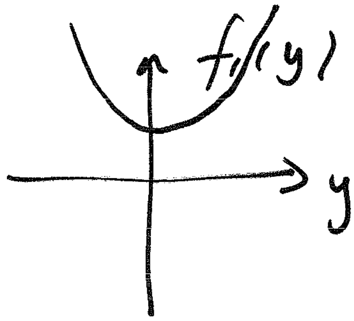
$$\frac{dy}{dt} = \mu + y^2$$

find the value(s) of μ where a bifurcation occurs and plot the bifurcation diagram.

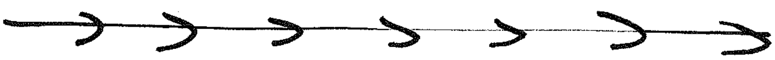
$$f_\mu(y) = \mu + y^2$$



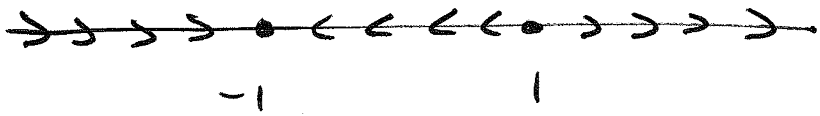
$\mu = 0$

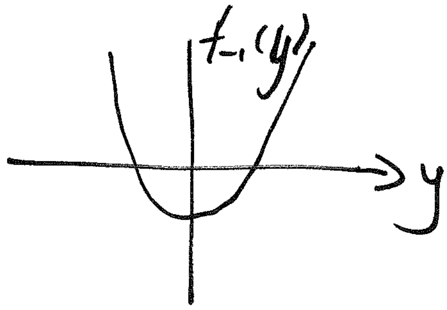


$\mu = 1$

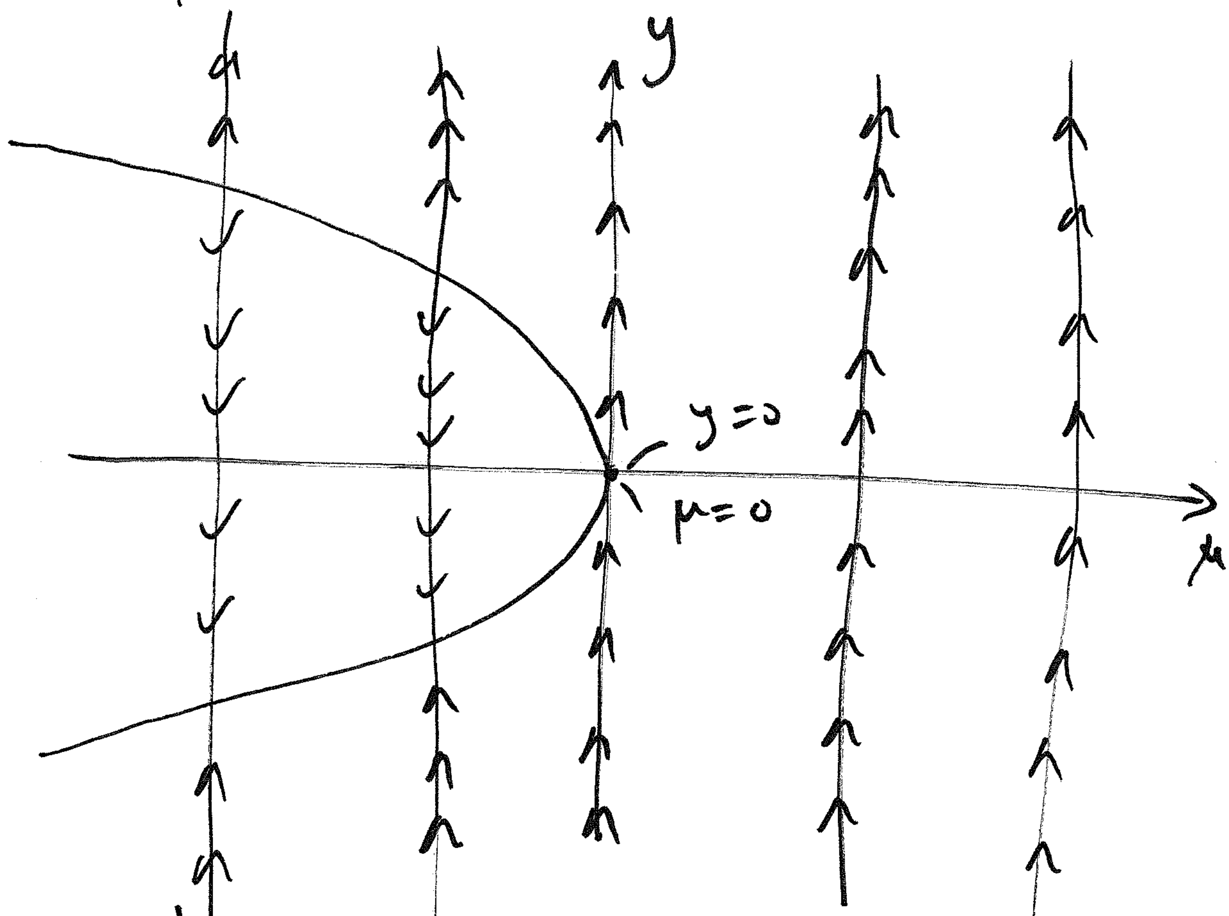


$\mu = -1$





Bifurcation



$$\frac{dy}{dt} = y^2 + \mu = 0 \Rightarrow y = \pm\sqrt{-\mu}$$

$(\mu = -y^2)$

Important ideas from today

A bifurcation occurs when a small change in parameter gives a qualitative change in the behaviour of solutions.

A bifurcation diagram is a picture which summarises the qualitative changes in behaviour that occur near a bifurcation.

Maths 260 Lecture 11

Topic for today

Bifurcations (continued)

Reading for this lecture

BDH Section 1.7

Suggested exercises

BDH Section 1.7: 11

Reading for next lecture

BDH Section 1.8

Today's handouts

Lecture 11 notes

We are interested in one-parameter families of autonomous DEs:

$$\frac{dy}{dt} = f_\mu(y).$$

We look for bifurcations, i.e., changes in the qualitative behaviour of solutions as the parameter μ is varied.

General result about bifurcations

Bifurcations usually do not happen, i.e., a small change in the parameter usually leads to only a small change in the behaviour of solutions.

To be precise, if

$$\frac{dy}{dt} = f_\mu(y)$$

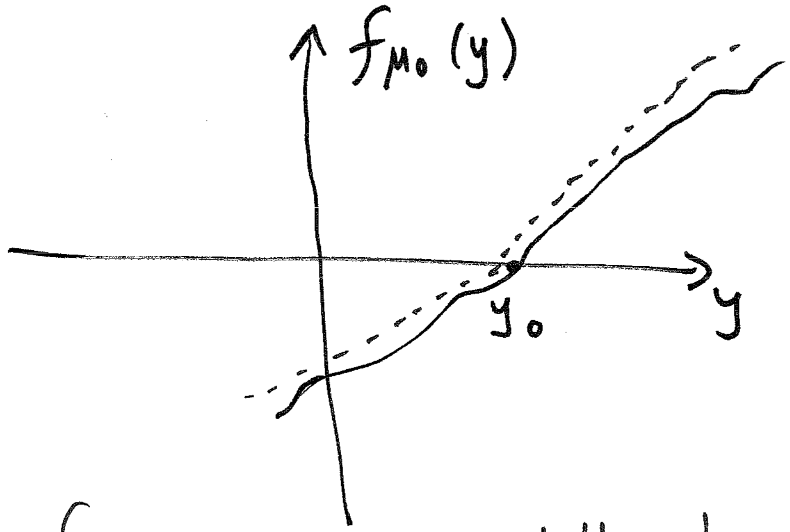
where $\partial f/\partial \mu$ and $\partial f/\partial y$ exist and are continuous for all values of μ and y, then a small change in μ gives a small change in the graph of $f_\mu(y)$.

Example: Suppose the DE

$$\frac{dy}{dt} = f_\mu(y) \quad (f_{\mu_0}(y_0) = 0)$$

with $\mu = \mu_0$ has a source at $y = y_0$ with $\boxed{df_{\mu_0}/dy > 0.}$ → condition for source

What is the effect on the qualitative behaviour of solutions of changing μ by a small amount?



- - - - $f_\mu(y)$ where μ is near μ_0

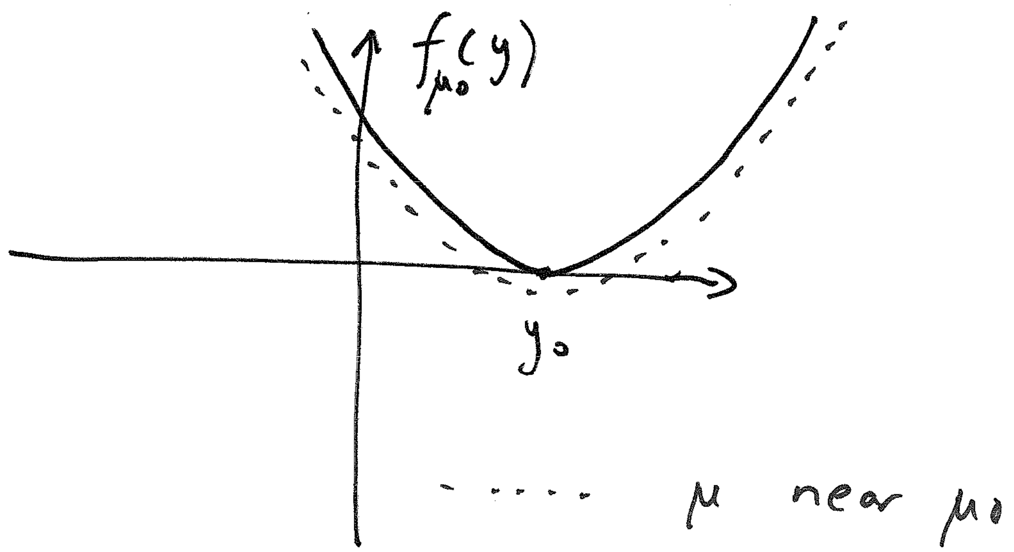
for - - - - still have a zero close to y_0 and it is still a source

A bifurcation where the number or type of equilibria changes can only occur at $\mu = \mu_0$ if

$$f_{\mu_0}(y_0) = 0 \text{ and } \frac{df_{\mu_0}}{dy}(y_0) = 0,$$

i.e., when the linearization theorem does not work.

e.g.

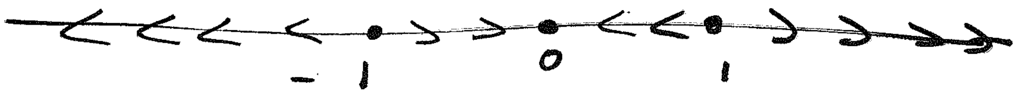


dotted lines and solid line different
solid has one zero - dotted has 2.

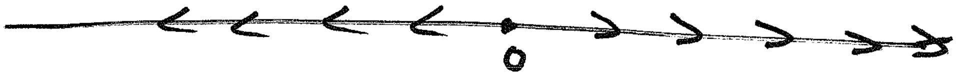
Example: Draw the bifurcation diagram for the family of equations

$$\frac{dy}{dt} = \mu y + y^3 \quad = f_\mu(y)$$

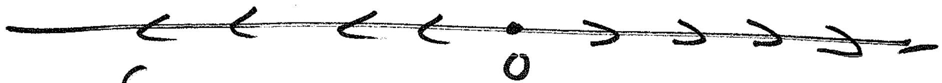
$\mu = -1$



$\mu = 0$

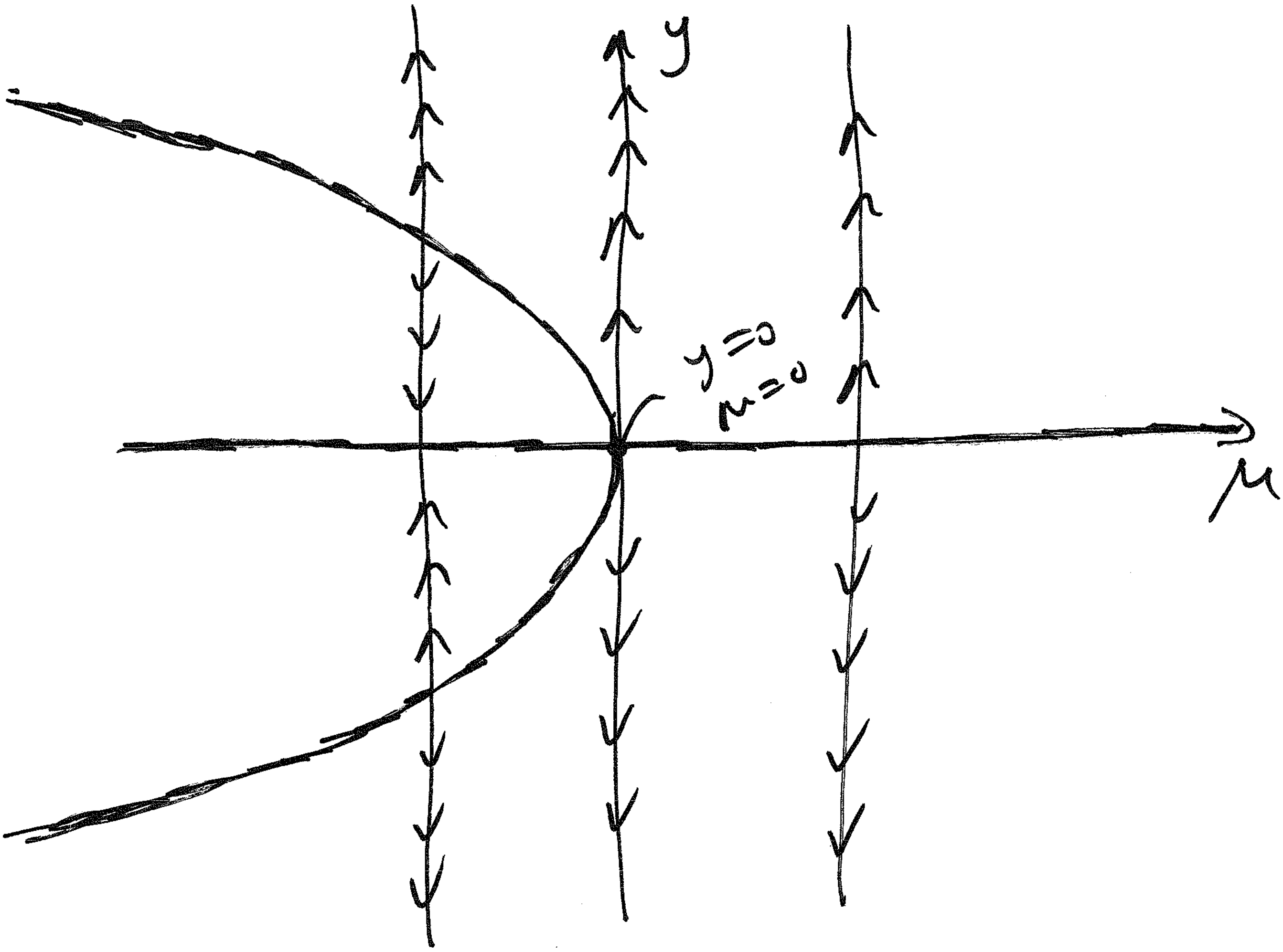


$\mu = 1$



$f_\mu(y) = 0 \Rightarrow y = 0$ or $y = \pm\sqrt{-\mu}$

$$\frac{dy}{dt} = \mu y + y^3$$



e.g. Equilibrium points $y = 0$

$$\underbrace{y = \pm\sqrt{-\mu}}_{\mu = -y^2}$$

Bifurcation at $\mu = 0$, $y = 0$.

If we solve " $f_\mu = 0$ & $\frac{\partial f_\mu}{\partial y} = 0$

$$f_\mu = \mu y + y^3 = 0 \quad (*)$$

$$\frac{\partial f_\mu}{\partial y} = \mu + 3y^2 = 0 \quad (**)$$

$(**) \Rightarrow \mu = -3y^2$ and substitute into $(*)$

$$-3y^2 \cdot y + y^3 = 0$$

$$\Rightarrow -2y^3 = 0$$

$$\Rightarrow y = 0$$

& substitute $y=0$ into $(*)$

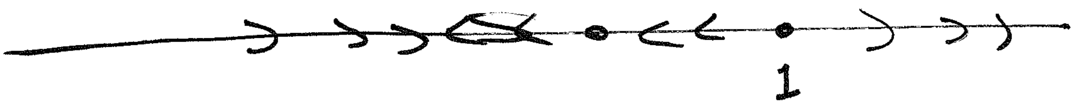
$$\mu = 0$$

i.e. bifurcation at $y=0$ & $\mu=0$.

Example: Draw the bifurcation diagram for the family of equations

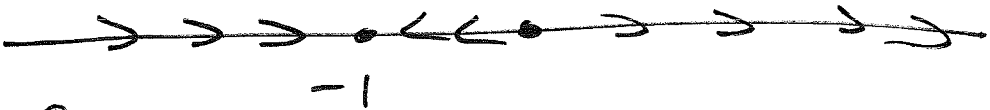
$$\frac{dy}{dt} = \mu y + y^2 = f_\mu(y)$$

Phase line $\mu = -1$



$\mu = 0$

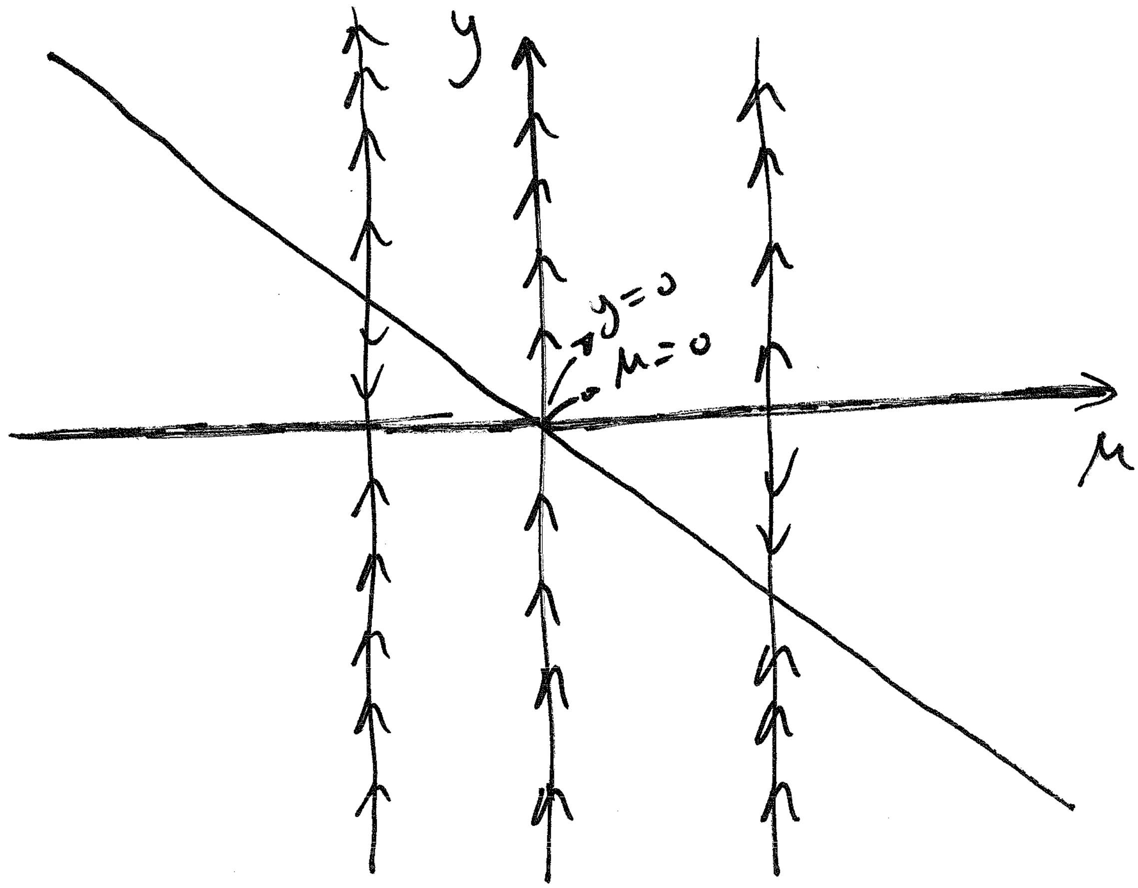
$\mu = 1$



$$\left(f_\mu(y) = 0 \Rightarrow y = 0, \; y = -\mu \right)$$

$$\frac{dy}{dt} = \mu y + y^2 = f_\mu(y)$$

$$f_\mu(y) = 0 \Rightarrow y = 0 \text{ , or } y = -\mu$$



Bifurcation at $\mu = 0$

(check solve $f_\mu(y) = 0 \Rightarrow \mu y + y^2 = 0$

$$\frac{\partial f_\mu}{\partial y} = 0 \Rightarrow \mu + 2y = 0$$

What is soln to $\mu y + y^2 = 0$ & $\mu + 2y = 0$

$\mu y + y^2 = 0 \Rightarrow y = 0$ or $y = -\mu$.

substitute $y = 0$ & $y = -\mu$ into $\mu + 2y = 0$

$\Rightarrow y = 0$ & $\mu = 0$

Important ideas from today

Bifurcations are special: a small change in parameter does not usually result in a qualitative change in the behaviour of solutions.

A bifurcation where the number or type of equilibria changes can only occur at $\mu = \mu_0$ if

$$f_{\mu_0}(y_0) = 0 \text{ and } \frac{df_{\mu_0}}{dy}(y_0) = 0.$$

Maths 260 Lecture 12

Topic for today

Linear differential equations

Reading for this lecture

BDH Section 1.8

Suggested exercises

BDH Section 1.8: 1, 3, 9, 13

Reading for next lecture

BDH Section 2.1

Today's handouts

Lecture 12 notes

Section 1.8: Linear Differential Equations

A first order DE is **linear** if it can be written in the form

$$\left.\frac{dy}{dt} = g(t)y + f(t)\right\}$$ compare to separable

where $g(t)$ and $f(t)$ are arbitrary functions of t.

Examples:

1. $\dfrac{dy}{dt} = y\cos t + t^2$

2. $y\dfrac{dy}{dt} = ty^2 + ty \Rightarrow \dfrac{dy}{dt} = ty + t$

3. $\left(t^2+1\right)\dfrac{dy}{dt} + 2ty - 1 = 0$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-2t}{t^2+1} + \frac{1}{t^2+1}$$

4. $\frac{dy}{dt} = ty\,(1-y)$ is nonlinear $\rightarrow$ (from $-ty^2$)

Linear means that the dependent variable y appears in the equation only to the first power.

Finding Solutions to linear DEs

First rewrite the DE as

$$\frac{dy}{dt} + a(t)\, y = f(t)$$

(where $a(t) = -g(t)$).

A clever trick:

Multiply through by $\mu(t)$, an unknown, non-zero function which will be determined later.

We have

$$\mu(t)\frac{dy}{dt} + \mu(t)\, a(t)\, y = \mu(t)\, f(t)$$

$(\mu(t) \neq 0)$

Now assume we can pick $\mu(t)$ so that

$$\mu(t)\frac{dy}{dt} + \mu(t)\, a(t)\, y = \frac{d}{dt}(\mu(t)\, y)$$

So

$$\frac{d}{dt}(\mu(t)\, y) = \mu(t)\, f(t)$$

Integrating both sides with respect to t:

$$\mu(t)y(t) = \int \mu(t)f(t)dt$$

$$\Rightarrow \boxed{y(t) = \frac{1}{\mu(t)}\int \mu(t)f(t)dt}$$

If we can find such a $\mu(t)$ and do the integration then we can find $y(t)$.

The function $\mu(t)$ is called an integrating factor.

Formula for Solution once we have $\mu(t)$

Finding the integrating factor, $\mu(t)$.

We want $\mu(t)$ such that

$$\mu(t)\frac{dy}{dt} + \mu(t)a(t)y = \frac{d}{dt}(\mu(t)y)$$
$$= \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$$

After cancelling terms, this is:

$$\mu(t)a(t)y = \frac{d\mu}{dt}y$$
$$\Rightarrow \boxed{\frac{d\mu}{dt} = \mu(t)a(t)}$$

(like $\frac{dy}{dt} = y\,a(t)$)

This is a separable DE for μ. Solve it:

$$\int \frac{d\mu}{\mu} = \int a(t)dt$$
$$\Rightarrow \ln|\mu| = \int a(t)dt \quad + c$$
$$\Rightarrow \mu(t) = \pm A\exp(\textstyle\int a(t)dt)$$

Different choice of the constant of integration will give different μ, but all choices give a valid integrating factor. Pick the easiest.

ie. set $A = 1$

$$\mu(t) = e^{\int a(t)\,dt}$$

Summarise (Dummies guide)

$$\frac{dy}{dt} + a(t)\,y = f(t).$$

1) Find $\mu = e^{\int a(t)\,dt}$ (do not need constant)

2) $y = \frac{1}{\mu(t)}\left(\int \mu(t) f(t)\,dt + c\right)$

put constant here

Summary of method

To find a solution to

$$\frac{dy}{dt} + a(t)y = f(t)$$

find the integrating factor:

$$\mu(t) = \exp\left(\int a\,(t)\,dt\right)$$

Then the solution is

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) f(t) dt$$

Example 1: Find a one-parameter family of solutions to

$$\frac{dy}{dt} = \frac{y}{t} + t^4 \quad , \quad t > 0$$

$$\frac{dy}{dt} - \frac{y}{t} = t^4$$

$$a(t) = \frac{-1}{t} \quad , \quad f(t) = t^4$$

$$\mu = e^{\int a(t)\,dt}$$

set $c = 0$

$$\Rightarrow \int a(t)\,dt = \int \frac{-1}{t}\,dt = -\ln|t|$$

$$\mu = e^{-\ln|t|} = \left(e^{\ln|t|}\right)^{-1} = \frac{1}{|t|}$$

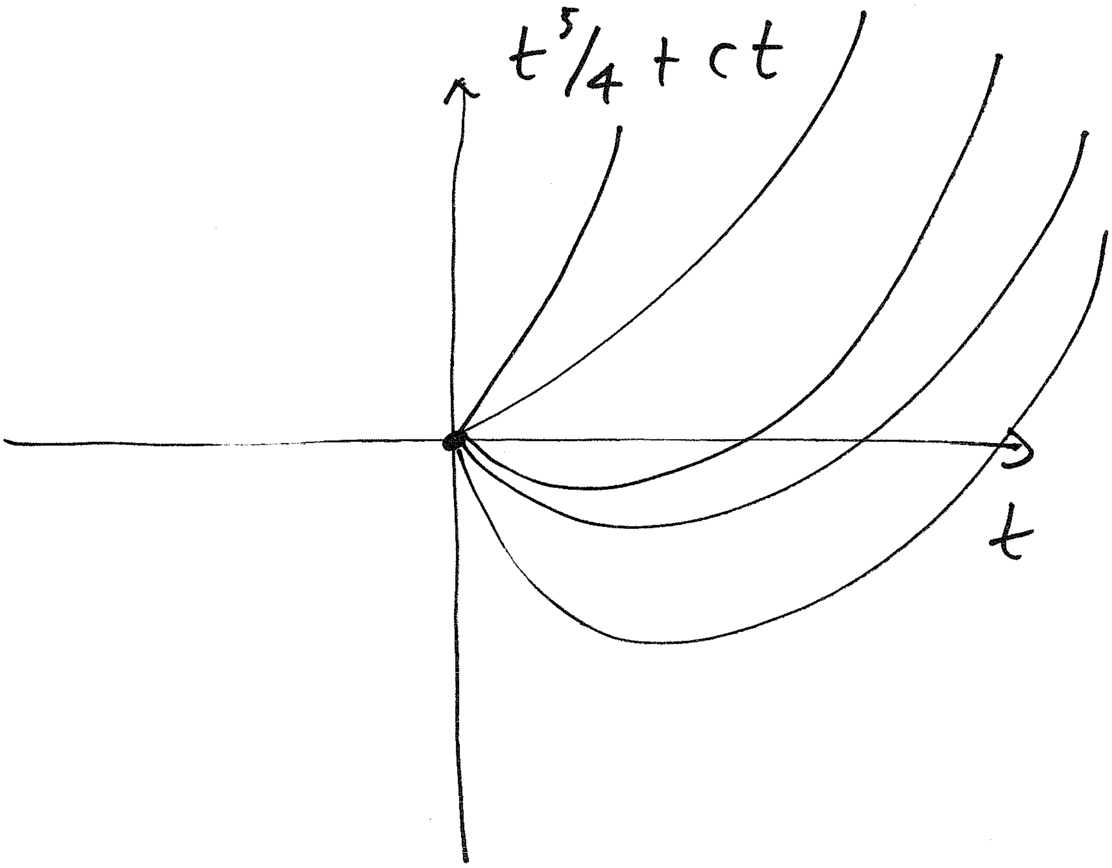
(tricky example and set $\mu = \frac{1}{t}$)

$$y(t) = \frac{1}{\mu}\left(\int \mu f(t)\,dt + c\right)$$

$$= t\left(\int \frac{1}{t} t^4\,dt + c\right)$$

$$= t\left(\frac{t^4}{4} + c\right) = \frac{t^5}{4} + ct$$

It's interesting to graph solutions for various values of the arbitrary constant.



Notice that in this case you can't always solve initial value problems with initial condition $y(t_0) = y_0$ and that when you can, the solution isn't unique. Is this what you expect (check the Existence and Uniqueness theorems).

Example 2: Find a solution to the IVP

$$\frac{dy}{dt} = -2y - 3t, \quad \boxed{y(0) = \frac{1}{2}}.$$

solve d.e. first and last step is to find c from $y(0) = 1/2$

$$\frac{dy}{dt} + 2y = -3t$$

$$a(t) = 2, \quad f(t) = -3t$$

1) $\mu = e^{\int a(t)dt} = e^{\int 2dt} = e^{2t}$

2) $y = \frac{1}{\mu}\left(\int \mu f(t)\,dt + c\right)$

$$= e^{-2t}\left(\int e^{2t}(-3t)dt + c\right)$$

$$= -3e^{-2t}\int te^{2t}dt + ce^{-2t}$$

$$= -3e^{-2t}\left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}\right) + ce^{-2t}$$

$$= -\frac{3}{2}t + \frac{3}{4} + ce^{-2t}$$

$y(0) = 3/4 + c = 1/2 \Rightarrow c = -1/4$

We find

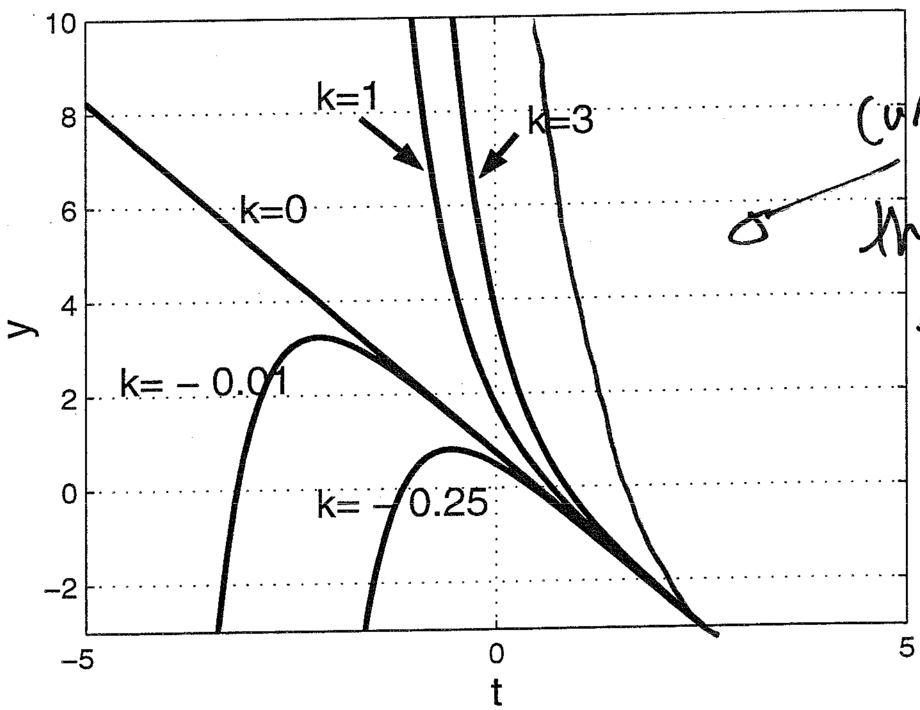
$$\frac{dy}{dt} = -2y - 3t$$

has a one-parameter family solutions

$$y(t) = -\frac{3}{2}t + \frac{3}{4} + ke^{-2t}.$$

The choice $y(0) = 1/2$ determines $k = -1/4$.

Some solutions to the DE including the solution to the IVP are plotted below.



curve through every point

Example 3: Find a solution to the DE

$$\frac{dy}{dt} = 1 + 2ty$$

$$\frac{dy}{dt} - 2ty = 1$$

$$\left(a(t) = -2t, \quad f(t) = 1 \right)$$

$$\mu = \cancel{\int} e^{\int a(t)dt} = e^{\int -2t\,dt} = e^{-t^2}$$

$$y = \frac{1}{\mu}\left(\int f(t)\mu(t)\,dt + c \right)$$

$$= e^{t^2}\left(\underbrace{\int e^{-t^2} dt}_{\text{we cannot calculate this integral}} + c \right)$$

$$=$$

$$\frac{dy}{dt} = 3y + 7 \quad \Big| \quad y(0) = 4$$

(also separable - see student loan)

$$\frac{dy}{dt} - 3y = 7$$

$$a(t) = -3, \quad f(t) = 7$$

$$\mu = e^{\int -3\,dt} = e^{-3t}$$

multiply by e^{-3t}

$$e^{-3t}\frac{dy}{dt} - 3e^{-3t}y = 7e^{-3t}$$

$$\frac{d}{dt}\left(e^{-3t}y\right) = 7e^{-3t}$$

$$e^{-3t}y = \left(\int 7e^{-3t}\,dt + c\right)$$

$$y = \frac{1}{e^{-3t}}\left(\int 7e^{-3t}\,dt + c\right)$$

$$y = \frac{1}{\mu}\left(\int \mu f(t)\,dt + c\right)$$

$$= \frac{1}{e^{-3t}}\left(\int 7e^{-3t}\,dt + c\right)$$

$$= e^{3t}\left(-\frac{7}{3}e^{-3t} + c\right)$$

$$y = -\frac{7}{3} + ce^{3t}$$

(soln to $\frac{dy}{dt} = 3y + 7$)

$y(0) = 4$

$$\Rightarrow -\frac{7}{3} + c = 4$$

$$\Rightarrow c = \frac{19}{3}$$

$$y = -\frac{7}{3} + \frac{19}{3}e^{3t}.$$