Maths 260 Lecture 13

Topic for today

Introduction to systems of differential equations

Reading for this lecture

BDH Section 2.1

Suggested exercises

BDH Section 2.1: 1-4, 9, 10

Reading for next lecture

BDH Section 2.2

Today's handouts

Lecture 13 notes

Chapter 2: Systems of First Order DEs

Section 2.1 Introduction: DEs that contain more than one dependent variable are known as systems of DEs.

Examples:

1.

$$\frac{dx}{dt} = -2x + 3y$$
$$\frac{dy}{dt} = -2y$$

2.

$$\frac{dx}{dt} = 10(y - x)$$
$$\frac{dy}{dt} = 28x - y - xz$$
$$\frac{dz}{dt} = -\frac{8}{3}z + xy$$

3.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - y + \mu\cos(t)$$

The purpose of today's lecture is to introduce some important ideas for the study of systems of DEs, but with formal definitions and other details mostly left to later lectures.

Mostly interested in systems of first order DEs. Write these in standard form:

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n)$$

$$\frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n)$$

The notation

$$\frac{dx_1}{dt} = \dot{x}_1, \ \frac{dx_2}{dt} = \dot{x}_2$$

is often used.

Solutions to systems of DEs

A **solution** to a system of n first order equations is a set of n functions that satisfy the differential equations.

Example: Determine which of the following pairs of functions is a solution to the system

$$\frac{dx}{dt} = -2x + 3y,$$
$$\frac{dy}{dt} = -2y.$$

1. $x(t) = -3te^{-2t}, \quad y(t) = -e^{-2t};$
2. $x(t) = 3e^{-2t}, \quad y(t) = 0;$
3. $x(t) = 3e^{-2t} + te^{-2t}, \quad y(t) = -e^{-2t}.$

Example: Model of two populations (predator/prey)

Let $R(t)$ = #prey ("Rabbits") in 1000's
Let $F(t)$ = #predators ("Foxes") in 1000's.
A possible model of change in the two populations is given by

$$\dot{R} = 0.4R - 0.1RF, \quad (1)$$
$$\dot{F} = -0.5F + 0.1RF, \quad (2)$$

with $R \geq 0$, $F \geq 0$.

Physical significance of terms on RHS

- The term $0.4R$ in (1) gives unlimited growth of prey if no predators exist.
- The term $-0.5F$ in (2) gives exponential decay in predator population if no prey exist.
- The term $-0.1RF$ in (1) models the negative effect on prey population of 'interactions' between prey and predators (i.e., predators eat prey and prey population decreases).
- The term $0.1RF$ in (2) models the positive effect on predator population of interactions between prey and predators (i.e., predators eat prey and predator population increases).

(. . . as long as prey have plenty to eat themselves!)

Solutions to the predator/prey system

Some special cases:

1. The pair of constant functions $R(t) = 0$, $F(t) = 0$ is an **equilibrium** solution.

2. Rewriting system as:

$$\begin{aligned}\dot{R} &= R(0.4 - 0.1F),\\ \dot{F} &= F(0.1R - 0.5),\end{aligned}$$

we see that $(R(t), F(t)) = (5, 4)$ is an equilibrium solution.

Physically, this tells us that a prey population of 5000 and a predator population of 4000 is perfectly balanced; neither population increases or decreases over time.

3. If $F(t) = 0$, then $\dot{F} = 0$ for all time, regardless of behaviour of x.

However, if $\dot{F} = 0$, then $\dot{R} = 0.4R$ so $R(t) = R_0 e^{0.4t}$ is a solution, i.e., if there are no predators, the prey population grows exponentially.

4. Similarly, if $R(t) = 0$, then $\dot{R} = 0$ for all time, regardless of behaviour of F.

However, if $\dot{R} = 0$, then $\dot{F} = -0.5F$ so $F(t) = F_0 e^{-0.5t}$ is a solution, i.e. if there are no prey, the predator population decreases exponentially.

Apart from special cases, don't (yet) have analytic or qualitative methods to investigate solutions to this DE. Continue to study this system using numerical methods (software on book CD). Details of numerical methods for systems are in a later lecture.

Representing solutions graphically

Can plot graphs of R and F as functions of t.

From the program on the book CD we see:
(i) If $R(0) = 5$, $F(0) = 4$, get equilibrium solutions as expected.

(ii) If $R(0) = 0$, $F(0) > 0$, or $R(0) > 0$, $F(0) = 0$, get exponentially decreasing or increasing solutions as expected.

(iii) All other solutions with $R(0) > 0$ and $F(0) > 0$ are periodic (same period for R and F).

Can plot solutions in **phase space**, i.e., for given t, plot the point $(R(t), F(t))$ in $R - F$-space.

As t varies, point will move and sweep out a curve in $R - F$ space. This curve is the **solution curve** in phase space.

From this program we see that

1. equilibrium solutions correspond to a single point in the phase space, i.e., solution curve is just a single point.
2. periodic solutions correspond to a closed curve in phase space.

Use arrow to show direction that move along solution curve as time increases.

Phase space (here: phase plane) is the higher dimensional equivalent of phase line. Solutions drawn in phase space don't show explicit values of t, just how the dependent variables change as t changes.

Different solution curves plotted on the same phase space picture give the **phase portrait** of the system. For example, the phase portrait for the predator/prey system is:

The aim of this section of Maths 260 is to develop qualitative, analytic and numerical methods for getting information about systems of differential equations.

Maths 260 Lecture 14

Topics for today

Direction fields and solutions
Equilibrium solutions
Using the tool *pplane* from Matlab

Reading for this lecture

BDH Section 2.2

Suggested exercises

BDH Section 2.2: 1st ed. 1, 3, 13, 17-20, 21-24, 29
2nd ed. 1, 3, 11, 13-16, 19, 27

Reading for next lecture

BDH Section 2.4

Today's handout

Lecture 14 notes

Section 2.2 Direction Fields

Directions fields are the analogue for systems of equations of slope fields.

Consider a system of two autonomous DEs:

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

Write

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

and

$$\mathbf{V}(\mathbf{Y}) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

Then the system written in vector form is

$$\frac{d\mathbf{Y}}{dt} = \mathbf{V}(\mathbf{Y})$$

$\mathbf{V}(\mathbf{Y})$ is known as a **vector field** i.e., it is a function that assigns a vector to each point of the (x, y)-plane.

Example

$$\frac{dx}{dt} = 0.5x - 0.4xy$$

$$\frac{dy}{dt} = -y + 0.2xy$$

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$V(Y) = \begin{pmatrix} 0.5x - 0.4xy \\ -y + 0.2xy \end{pmatrix}$$

Sometimes write

$$\mathbf{Y} = (x, y)$$

and

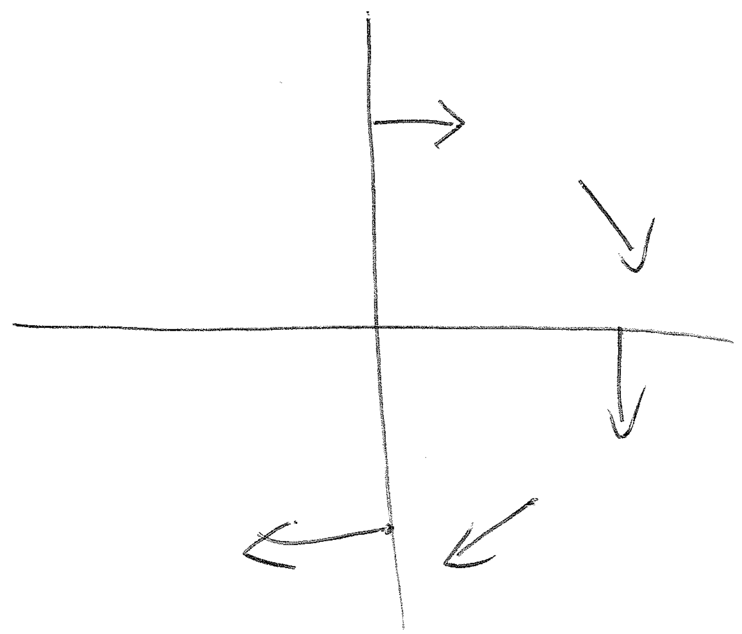
$$\mathbf{V} = (f(x, y), g(x, y))$$

Plotting vector fields:

At point $\mathbf{Y_0} = (x_0, y_0)$ in the x, y-plane draw the vector $\mathbf{V}(\mathbf{Y_0})$ with the base of vector at $\mathbf{Y_0}$ and with arrow showing direction of vector.

Example:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

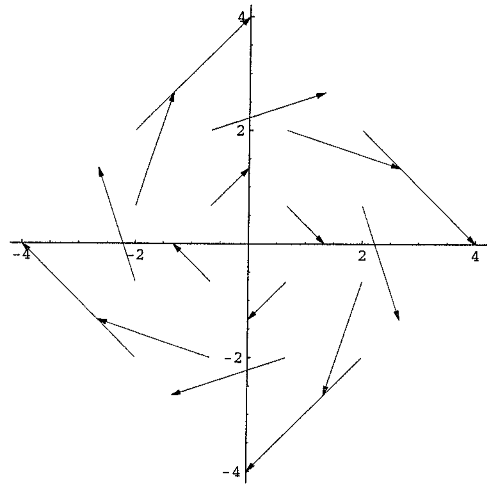


Problem with plotting vector fields: vectors can cross, which makes a big mess. For example, for the system

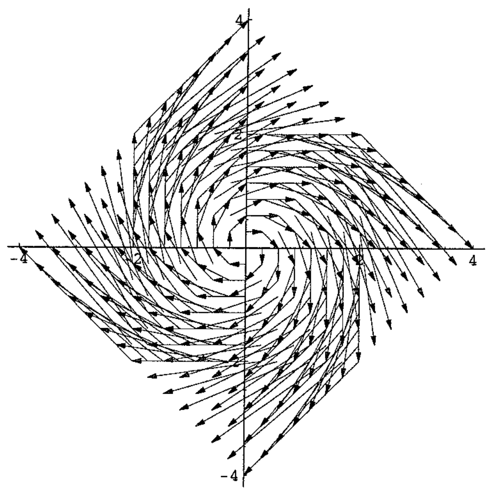
$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x$$

Vector field with selected vectors:

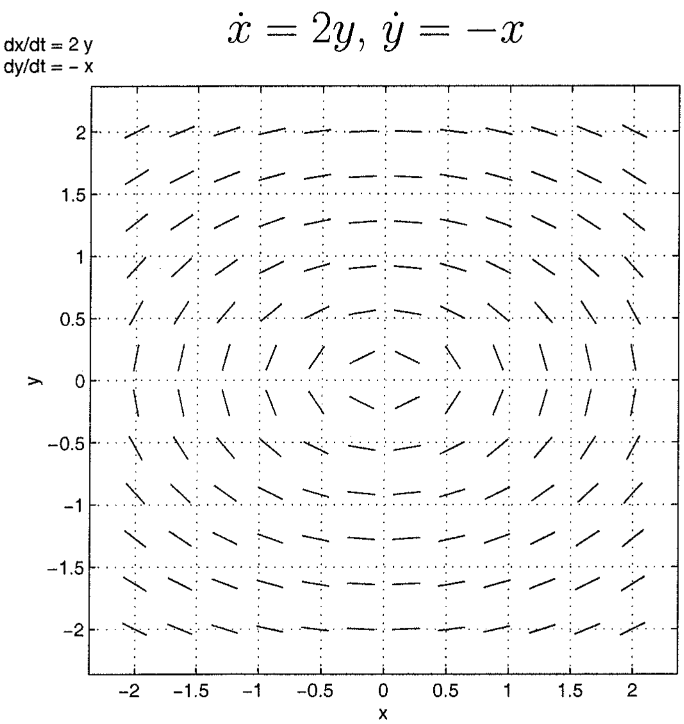


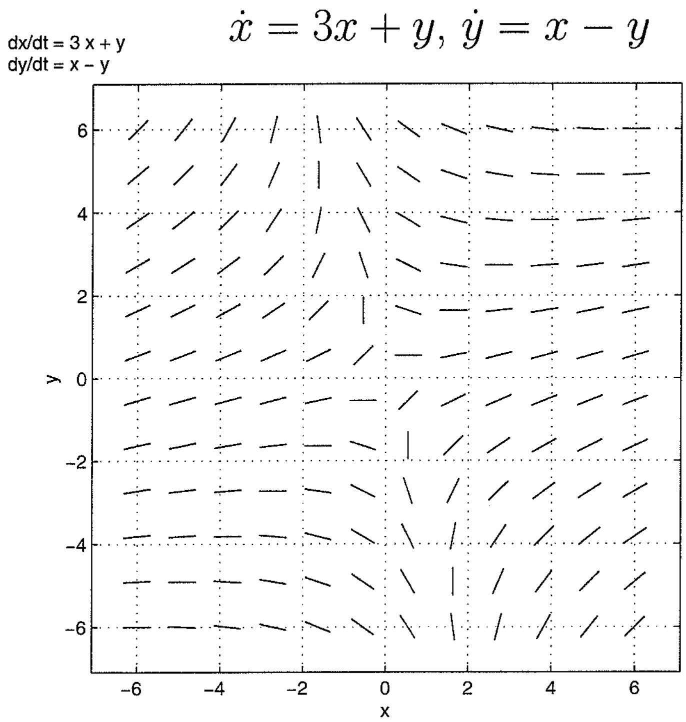
Vector field with more vectors:



Avoid these problems by plotting **direction field**, i.e., vectors with same direction as in vector field but scaled to a uniform length. Arrowheads may or may not be shown.

The following pictures show the direction fields for some systems.





We can use *pplane* from *Matlab* to plot direction fields. See the lab handout for details on using *pplane*.

Sketching solutions to systems:

Consider a system of DEs

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}), \quad \mathbf{Y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{1}$$

A solution is a vector of functions $\mathbf{Y}(t)$ and corresponds to a curve in phase space, parameterized by time (i.e., vary t to move along curve).

The vector

$$\left.\frac{d\mathbf{Y}}{dt}\right|_{t=t_0}$$

is tangent to curve of $\mathbf{Y}(t)$ at $t = t_0$.

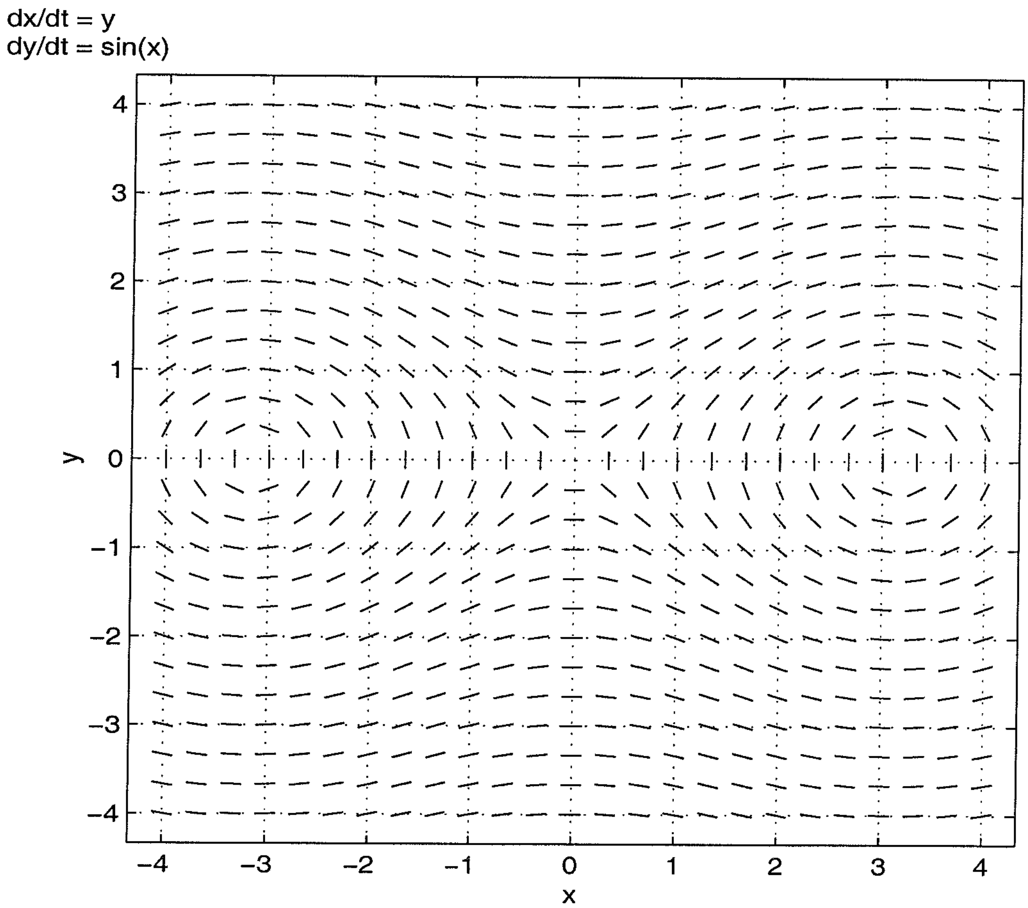
Thus, equation (1) says that vectors in direction field are tangent to solutions of DE. So to sketch solution curves to DE (1),

1. plot direction field, then
2. starting at some initial point, sketch a smooth curve that follows vectors in direction field.

Example: Sketch some representative solutions for the system

$$\dot{x} = y$$
$$\dot{y} = \sin(x)$$

The direction field is given below.



Equilibrium solutions

The point $\mathbf{Y}_0$ is an *equilibrium point* for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{V}(\mathbf{Y})$$

if $\mathbf{V}(\mathbf{Y}_0) = 0$.

If $\mathbf{Y}_0$ is an equilibrium point, then the constant function $\mathbf{Y}(t) = \mathbf{Y}_0$ is a solution of the system.

Example 1

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 2y + x \end{aligned}$$

$2x + y = 0 \Rightarrow y = -2x$

$2y + x = 0 \Rightarrow y = -\frac{x}{2}$

$\Rightarrow -2x = -\frac{x}{2} \Rightarrow x = 0$

eqilibrium point (eq. pt. $(0,0)$)

Example 2

$$\dot{x} = x + y$$
$$\dot{y} = y\,(2 - x)$$

$x + y = 0 \Rightarrow y = -x$

$y\,(2-x) = 0$

$\Rightarrow -x\,(2-x) = 0$

$\Rightarrow x = 0$ or $x = 2$

eq. pt $(0,0)$ $(2,-2)$

Behaviour of solutions near equilibria can be observed with *pplane*. Note that

1. direction of vectors in direction field changes dramatically near an equilibirum point, and
2. solutions passing near an equilibrium go very slowly (because all components of vector field $\to 0$ near an equilibrium).

Maths 260 Lecture 15

Topics for today

Numerical methods for systems
Existence and Uniqueness Theorem for systems

Reading for this lecture

BDH Section 2.4

Suggested exercises

BDH Section 2.4: 7, 8, 9, 10

Reading for next lecture

BDH Section 2.3, pp 175–178 (1st ed) 185–188 (2nd ed); Section 3.1

Today's handouts

Lecture 15 notes

Section 2.3 Numerical Methods for Systems

Numerical methods used for first order equations can be generalised to systems of first order equations.

Example: Euler's Method for systems

Given the IVP

$$\frac{dx}{dt} = f(t, x, y),$$
$$\frac{dy}{dt} = g(t, x, y),$$

with $x(t_0) = x_0$ and $y(t_0) = y_0$, then Euler's Method calculates the approximate solution at $t_1 = t_0 + h$ to be

$$x(t_0 + h) \approx x_0 + hf(t_0, x_0, y_0),$$
$$y(t_0 + h) \approx y_0 + hg(t_0, x_0, y_0)$$

Can repeat to find approximation after n steps.

Example: Use Euler's method with $h = 0.1$ to calculate an approximate solution at $t = 0.2$ to the IVP

$$\frac{dx}{dt} = t + y,$$
$$\frac{dy}{dt} = y^2 - x$$

$x(0) = 1, y(0) = 0.$

$x_0 = 1 \quad , \quad y_0 = 0 \quad , \quad t_0 = 0$

$x(0.1) \approx x_1 = x_0 + h(t_0 + y_0)$

$= 1$

$y(0.1) \approx y_1 = y_0 + h(y_0^2 - x_0)$

$= 0 - 0.1 = -0.1$

$x(0.2) \approx x_2 = x_1 + h(t_1 + y_1)$

$= 1 + 0.1(0.1 - 0.1)$

$= 1$

$y(0.2) \approx y_2 = y_1 + 0.1(y_1^2 - x_1)$

$= -0.1 + 0.1(0.01 - 1) = -0.1998$

Vector Form of Euler's Method

Let

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

let

$$\mathbf{F}(t, \mathbf{X}) = \begin{pmatrix} f_1(t, x_1, x_2, \ldots, x_n) \\ f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \ldots, x_n) \end{pmatrix},$$

and let

$$\mathbf{X}_0 = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}.$$

Then the Euler approximation to the solution of the IVP

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(t, \mathbf{X}), \quad \mathbf{X}(t_0) = \mathbf{X}_0$$

at $t_0 + h$ is

$$\mathbf{X}(t_0 + h) \approx \mathbf{X}_0 + h\mathbf{F}(t_0, \mathbf{X}_0)$$

It can be proved that Euler's method for systems is first order, i.e., the error in the ith component of $\mathbf{X}$ is

$$|E_i(h)| \approx k_i h$$

in the limit of small h, where k_i is a constant.

Thus, halving step size will approximately halve the error in the estimated value of each component in $\mathbf{X}$.

Improved Euler and 4th order Runge-Kutta methods also generalize to systems and are order 2 and 4 respectively.

Existence and Uniqueness Theorem for systems

Consider the IVP

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}), \quad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

If $\mathbf{F}$ is continuous and has continuous first partial derivatives then there is an $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $\mathbf{Y}(t)$ is a solution to the IVP.

For t in this interval, the solution is unique.

Interpretation of EU Theorem

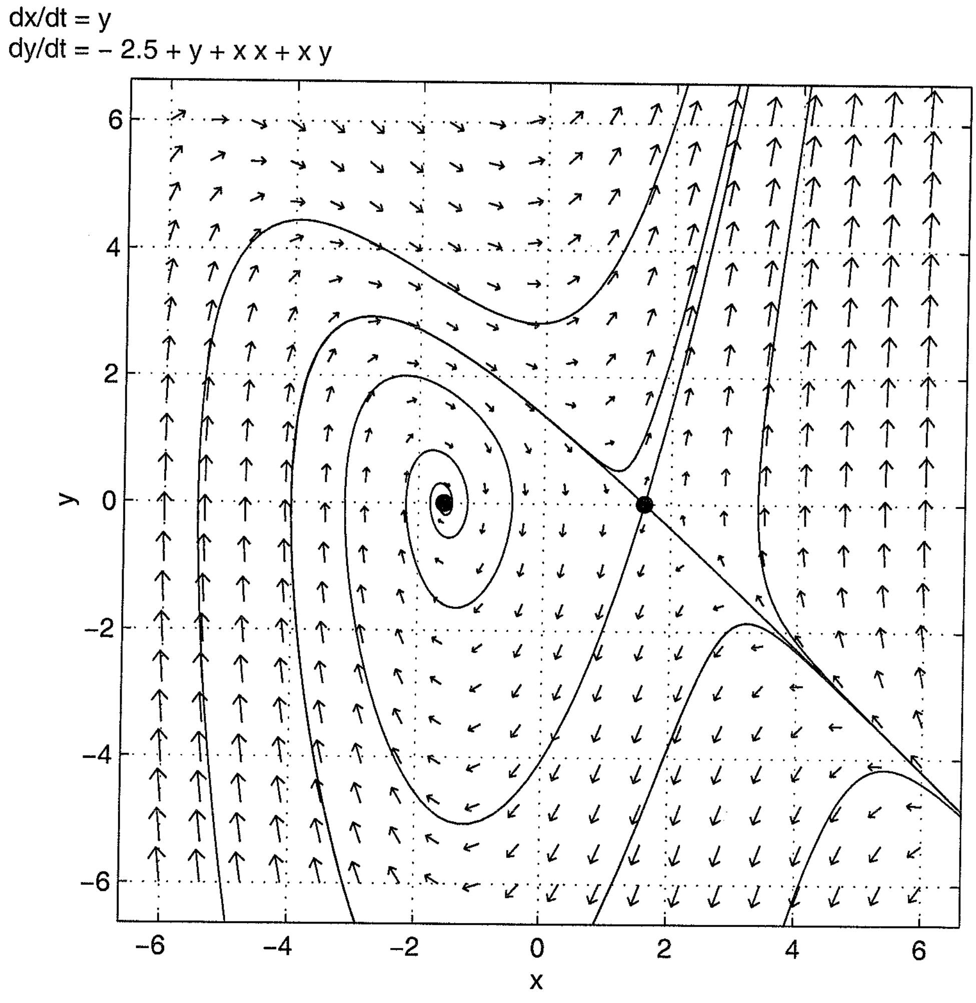
If a system of equations is 'nice' enough, a solution to an IVP exists and is unique.

In particular, two different solutions cannot start at the same t at the same point in phase space.

For autonomous systems, two different solutions that start at the same place in phase space but at different times will correspond to the same solution curve, i.e., solution curves cannot meet or cross in phase space.

Example: The phase portrait for the following differential equation is given below. It looks as though different solution curves meet/cross but EU Theorem ensures they do not.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2.5 + y + x^2 + xy$$



No such guarantee exists for solution curves of non-autonomous systems; solution curves for non-autonomous systems frequently cross in phase space.

Important ideas from today

- Numerical methods work for systems of DEs in a similar way as for single equations.
- 'Nice' IVPs have unique solutions.
- Solution curves for autonomous systems do not cross or meet in phase space.