Maths 260 Lecture 27

Topic for today

Higher order differential equations

Reading for this lecture

BDH Section 3.6

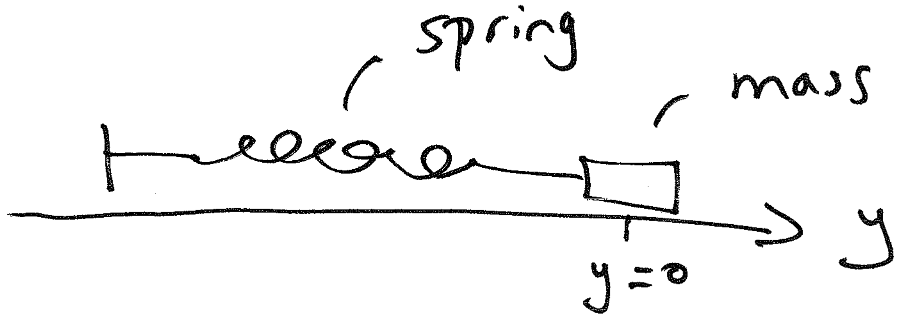
Today's handout

Lecture 27 notes

Section 3: Higher Order Differential Equations

Example: Modelling a mass/spring system

We wish to model the motion of an object that is attached to a spring and slides in a straight line on a table.



Let $y(t)$ =position of object at time t with $y = 0$ corresponding to the spring being neither stretched nor compressed.

Main idea from physics :
Newton's second law says

$$\text{mass} \times \text{acceleration} = \text{sum of forces}$$

$$m \frac{d^2y}{dt^2} = \sum \text{forces}$$

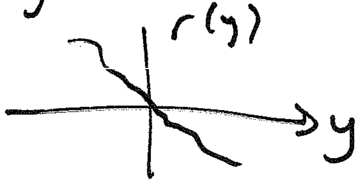
Typical forces on the object that we might consider are

1. restoring force (spring does not like to be compressed or stretched);

$$r(y) \quad \& \quad r(0) = 0$$

$$\& \quad \text{if } y < 0 \quad r(y) > 0$$

$$\text{if } y > 0 \quad r(y) < 0$$

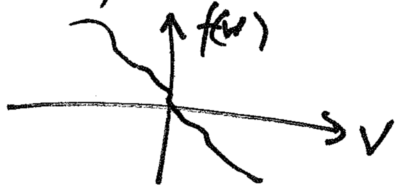


2. frictional forces;

$$\text{proportional to velocity} = \frac{dy}{dt} = v$$

$$f(0) = 0, \quad \text{if } v > 0 \quad f(v) < 0$$

$$\& \text{ if } v < 0 \quad f(v) > 0$$



3. external forcing.

Arbitrary function $g(t, y)$

Substituting into Newton's law, we get

$$m\frac{d^2y}{dt^2} = r(y) + f(v) + g\,(t, y)$$

where

- $r(y)$ represents the restoring force at position y
- $f(v)$ gives the frictional forces at velocity $v = \frac{dy}{dt}$
- $g\,(t, y)$ models any external forcing,
- m is the mass of the object attached to the spring.

A common case assumes

- linear restoring force (i.e., $r(y) = -ky$ for some constant $k > 0$),
- linear damping (i.e., $f(v) = -bv$ for some constant $b > 0$),
- no spatial dependence in the forcing (i.e., g a function of t but not y).

The first two assumptions may be valid if y and $v = \frac{dy}{dt}$ remain small.

We can write this case as

$$\frac{d^2y}{dt^2} + \underbrace{\frac{b}{m}\frac{dy}{dt}}_{\text{damping}} + \frac{k}{m}y = \frac{1}{m}g(t)$$

This differential equation is an example of a *higher order differential equation*, i.e., a DE involving derivatives of second or higher order.

Other examples of higher order DEs :

1.
$$\frac{d^2\theta}{dt^2} + c_1\frac{d\theta}{dt} + c_2 \sin\theta = 0$$

2.
$$\frac{d^3y}{dt^3} - 2y\left(\frac{d^2y}{dt^2}\right)^2 + \frac{dy}{dt} = \sin t$$

3.
$$\left.\begin{aligned} \frac{dx}{dt} &= 2x + y \\ \frac{d^2y}{dt^2} + \left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + 3x &= 0 \end{aligned}\right\} \text{system of equations}$$

We can usually convert a higher order DE into an equivalent system of first order DEs. To do so, define new dependent variables as in the following examples.

Examples:

1.

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad (*)$$

Introduce new variables, x_1 & x_2

$$x_1 = y$$

$$x_2 = \frac{dy}{dt}$$

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{d^2y}{dt^2} = -\frac{k}{m}y = -\frac{k}{m}x_1$$

i.e.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (**)$$

$(*)$ & $(**)$ are equivalent

i.e. they have same soln.

2.

$$\frac{d^3x}{dt^3} + 2\left(\frac{dx}{dt}\right)^2 = \sin t$$

New variables $\quad x_1 = x, \quad x_2 = \frac{dx}{dt}, \quad x_3 = \frac{d^2x}{dt^2}$

$$\frac{dx_1}{dt} = x_2 \qquad \Big| \qquad \cancel{\frac{dx_2}{dx}} \; \frac{dx_2}{dt} = x_3$$

$$\frac{dx_3}{dt} = \frac{d^3x}{dt^3} = -2\left(\frac{dx}{dt}\right)^2 + \sin t$$

$$= -2(x_2)^2 + \sin t$$

i.e. system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_3 \\ \frac{dx_3}{dt} &= -2(x_2)^2 + \sin t \end{aligned}\right\}$$

not a matrix system because it is not linear

Saying that a system of DEs is *equivalent* to a higher order DE means that if we know a solution to the system we can find one for the higher order equation, and vice versa.

Example

The function

$$y_1(t) = \sin\sqrt{\frac{k}{m}}t$$

is a solution to

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

What are eigenvalues / eigenvectors of

$\begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix}$, eigenvalues $\lambda = \pm i\sqrt{k/m}$

$\lambda = i\sqrt{k/m} \leftrightarrow v = \begin{pmatrix} 1 \\ +i\sqrt{k/m} \end{pmatrix}$

$$Y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix} = C_1 \cancel{\begin{pmatrix} \cos\sqrt{k/m} \\ -\sin\sqrt{k/m} \end{pmatrix}} + C_2 \cancel{\begin{pmatrix} \sin\sqrt{k/m} \\ \cos\sqrt{k/m} \end{pmatrix}}$$

$$= C_1 \begin{pmatrix} \cos\sqrt{k/m}\,t \\ -\sqrt{k/m}\sin\sqrt{k/m}\,t \end{pmatrix} + C_2 \begin{pmatrix} \sin\sqrt{k/m}\,t \\ \sqrt{k/m}\cos\sqrt{k/m}\,t \end{pmatrix}$$

The pair of functions

$$\left(y_1(t), \frac{dy_1}{dt} = v_1(t)\right) = \left(\sin\sqrt{\frac{k}{m}}t, \sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t\right)$$

is a solution to the equivalent system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -\frac{k}{m}y$$

$$\frac{dx_1}{at} = x_2$$

$$\frac{dx_2}{dt} = -\frac{k}{m}y$$

To determine the behaviour of solutions of a higher order DE we can rewrite the DE as the equivalent first order system.

Then we can study the system using the numerical methods and qualitative techniques (e.g., sketching solutions via phase plane methods) already learnt. We can also use results like the Existence and Uniqueness Theorem.

However, in some special cases, it is convenient to study the original higher order equation directly.

For example, convenient analytic techniques exist for solving linear higher order equations (see next few lectures).

§3.2 Linear, Constant Coefficient, Higher Order DEs

A differential equation of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

where all a_i are constant, and $a_n \neq 0$, is called an nth order, linear, constant coefficient DE.

Example :

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

Could solve this by converting to a system, then finding eigenvalues and eigenvectors etc.

In this section, find a short cut for solving equations of this form.

For previous example, equivalent system is:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} y \\ z \end{pmatrix}$$

e.g. $= \begin{pmatrix} e^{\lambda t} \cdot 3 \\ e^{\lambda t} \cdot 7 \end{pmatrix}$

Expect solutions of the form $\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$.
First component of such a $\mathbf{Y}$ is $y(t) = ce^{\lambda t}$, for some constant c.

Hence, guess a solution to the higher order DE of the form $y = e^{\lambda t}$ where λ is to be determined.

Substitute this candidate solution into our DE:

$$y = e^{\lambda t}, \quad \frac{dy}{dt} = \lambda e^{\lambda t}, \quad \frac{d^2y}{dt^2} = \lambda^2 e^{\lambda t}$$

substitute into

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

$$\Rightarrow \lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6e^{\lambda t} = 0$$

$$\Rightarrow (\lambda^2 + 5\lambda + 6)e^{\lambda t} = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 6 = 0 \quad (\text{since } e^{\lambda t} \neq 0)$$

$$\Rightarrow \lambda = -2 \text{ or } \lambda = -3$$

To find a solution to the associated system,

We have found two solutions

$$y = e^{-2t} \quad \& \quad y = e^{-3t}$$

where -2 & -3 were solns of

$$\lambda^2 + 5\lambda + 6 = 0$$

$$\left(\text{recall} \quad \frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0\right)$$

general solution is

$$y = c_1 e^{-2t} + c_2 e^{-3t}$$

This is exactly what we would have got by using eigenvalues and eigenvectors to solve the system directly.

This 'guessing' method is usually shorter than solving the system directly.

Summary:

A higher order differential equation can usually be rewritten as an equivalent system of first order differential equations. Solutions can then be investigated using the methods (qualitative, analytic, numerical) already studied for systems.

However, in the case of linear, constant coefficient higher order equations it is usually possible and quicker to find analytic solutions directly. The 'guessing' method we use will be formalised in the next lecture.

Maths 260 Lecture 30

Topic for today

Linear, constant coefficient, higher order DEs
IVPs for higher order DEs
The harmonic oscillator

Reading for this lecture

BDH Section 3.6 again

Suggested exercises

BDH Section 3.6; 1,3,5,7,9,11

Reading for next lecture

BDH Sections 4.1, 4.2

Today's handout

Lecture 30 notes

More on Linear, Constant Coefficient, Higher Order DEs

Consider the differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

Let $y_1(t)$, $y_2(t), \ldots, y_n(t)$ be n linearly independent solutions of the DE. Then

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t)$$

for arbitrary constants c_i, is called the **general solution** to the DE. Every solution to the DE can be written in this form by picking the c_i appropriately.

Example: Find the general solution to the differential equation

$$2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 0 \qquad (*)$$

(assume soln $y = e^{\lambda t}$) ← can omit

$\Rightarrow$ polynomial equation

$$2\lambda^2 + 5\lambda + 3 = 0$$

$$(2\lambda + 3)(\lambda + 1)$$

$$\lambda = -\frac{3}{2}, \quad \lambda = -1$$

general solution to (*)

$$y = c_1 e^{-3/2 t} + c_2 e^{-t}$$

Example: Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0$$

poly. eqn. $\lambda^2 + 4\lambda + 5 = 0$

$$\lambda = -2 \pm i$$

A soln is $y = e^{-2t+it}$

$$= e^{-2t}(\cos t + i\sin t)$$

$$= e^{-2t}(\cos t + i\sin t)$$

As before both real & imaginary parts must be solutions.

ie. Two linearly independent solutions are

$$y = e^{-2t}\cos t$$

& $y = e^{-2t}\sin t$

general soln is

$$y = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$$

Example: Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$

poly. eqn. $\lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow$ ~~$\lambda = 2$~~ $\lambda = -2$ repeated

one soln is $y = e^{-2t}$

a second soln is $y = te^{-2t}$

$y = c_1 e^{-2t} + c_2 te^{-2t}$

General Method:

To find the general solution to

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

1. Write down the **characteristic polynomial**:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots a_1 \lambda + a_0 = 0$$

   and find n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ (some may be repeated or complex).

   All functions of the form $e^{\lambda_i t}$, where λ_i is a root of the characteristic polynomial, will be solutions to the DE.

2. If all roots are distinct, can construct the general solution by taking a linear combination:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}$$

$\lambda_i = (\alpha + i\beta)$

   (converting to real form if necessary).

$e^{(\alpha + i\beta)t} = e^{(\alpha + i\beta)t} = e^{\alpha t} \cos\beta t$

$\& \; e^{\alpha t} \sin\beta t$

generally root repeated twice
& only need first two terms

3. If a root (say λ_i) is repeated k times, then the functions

$$\boxed{e^{\lambda_i t}, te^{\lambda_i t}}, t^2 e^{\lambda_i t}, \ldots, t^{k-1} e^{\lambda_i t}$$

are linearly independent solutions and we can use a linear combination of these in the general solution.

Remember that the general solution to an nth order linear, constant coefficient DE contains n arbitrary constants and n linearly independent solutions.

Example: Find the general solution to

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$$

char. poly $\lambda^3 + 3\lambda^2 + 2\lambda = 0$

$\Rightarrow \quad \lambda(\lambda^2 + 3\lambda + 2) = 0$

$\lambda = 0, \quad \lambda = -2, \quad \lambda = -1$

g.s. $y = c_1 e^{0t} + c_2 e^{-2t} + c_3 e^{-t}$

$\quad = c_1 + c_2 e^{-2t} + c_3 e^{-t}$

Example: Find the general solution to

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} = 0$$

char poly $\quad \lambda^3 + \lambda = 0$

$\cancel{\lambda = 0, \lambda}$

$$\lambda(\lambda^2 + 1) = 0$$

$$\lambda = 0, \ \lambda = \pm i \ (\alpha = 0, \beta = 1)$$

g.s. $\quad y = c_1 e^{0t} + c_2 e^{0t} \cos t + c_3 e^{0t} \sin t$

$$= c_1 + c_2 \cos t + c_3 \sin t$$

IVPs for Higher Order DEs

Consider a higher order DE such as

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

with associated system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{Y}$$

where $\mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$ and $v = \dfrac{dy}{dt}$.

To define an IVP for the system we specify an initial condition

$$\mathbf{Y}(t_0) = \mathbf{Y}_0 = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

i.e., $y(t_0) = y_0$ and $v(t_0) = \dfrac{dy}{dt}(t_0) = v_0$.

The equivalent IVP for the original higher order DE therefore has **two** initial conditions: $y(t_0) = y_0$ and $\dfrac{dy}{dt}(t_0) = v_0$.

More generally, an nth order IVP is formed from an nth order DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

together with n conditions

$$\begin{aligned} y(t_0) &= y_0, \\ \frac{dy}{dt}(t_0) &= y_1, \\ &\vdots \\ &\vdots \\ \frac{d^{n-1} y}{dt^{n-1}}(t_0) &= y_{n-1}. \end{aligned}$$

Example: Find a solution to the IVP

$$y'' - 2y' + 10y = 0 \qquad (*)$$

where $y(0) = 0$, $y'(0) = -2$.

Here (and elsewhere) $y' \equiv \dfrac{dy}{dt}$, $y'' = \dfrac{d^2y}{dt^2}$

First

Find general soln to $(*)$

char poly $\lambda^2 - 2\lambda + 10 = 0$

$$\lambda = 1 \pm 3i \quad (\alpha = 1, \beta = 3)$$

g.s. $y = c_1 e^t \cos 3t + c_2 e^t \sin 3t$

Now we find c_1 & c_2 from initial conditions

$y(0) = c_1 = 0$

$y'(t) = c_2 e^t \sin 3t + \cancel{Be^t}$
$\qquad + 3c_2 e^t \cos 3t$

$y'(0) = 3c_2 = -2$, $c_2 = -\frac{2}{3}$

$y(t) = -\frac{2}{3} e^t \sin 3t$

Solve $\quad y'' + 5y' + 6y = 0$

$$y(0) = 0$$

$$y'(0) = 1$$

g.s $\quad y = c_1 e^{-2t} + c_2 e^{-3t}$

I.V.P $\quad y = -e^{-2t} + e^{-3t}$

Solve $\quad y'' + 2y' + y = 0$

$$y(0) = 1, \quad y'(0) = 1$$

g.s $\quad y = c_1 e^{-t} + c_2 t e^{-t}$

~~I.V.P = $y = e^{-t} +$~~

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$$

$$y(0) = c_1 = 1$$

$$y'(0) = -c_1 + c_2 = 1$$

$$c_2 = 2$$

$$y = e^{-t} + 2te^{-t}$$

The Harmonic Oscillator

Consider the second order, linear, constant coefficient DE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

where $m, k > 0$, $b \geq 0$.

A physical system modelled by this equation is called a **harmonic oscillator**.

For instance, the mass/spring system considered in the last lecture is a harmonic oscillator if we assume linear damping and restoring forces, and no external forcing.

Can now completely classify the different types of solution to this problem.

Note that the equivalent system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic polynomial is

$$m\lambda^2 + b\lambda + k = 0 \qquad (m, b, k > 0)$$

which has roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, \lambda_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

and the general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

There are four different cases, depending on the size of b, the damping coefficient.

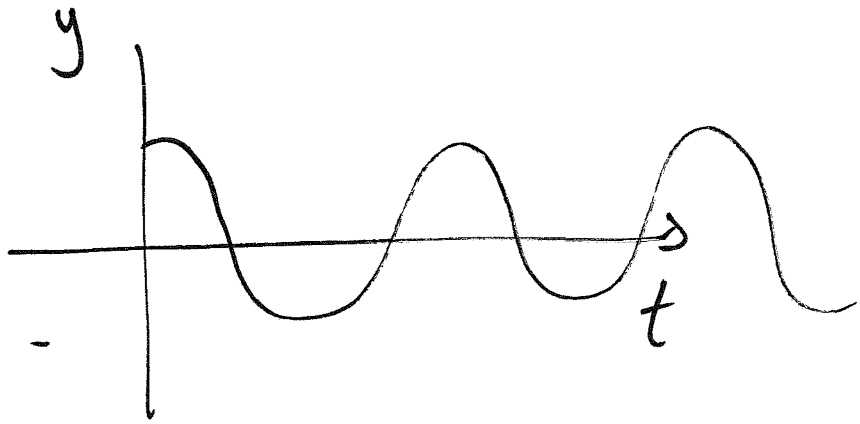
Case 1: $b = 0$ (no damping)

$$\lambda_1 = i\frac{\sqrt{4mk}}{2m} = i\sqrt{k/m}$$

$$\left(\lambda_2 = -i\sqrt{k/m}\right)$$

$$y = C_1 \cos\sqrt{k/m}\,t + C_2 \sin\sqrt{k/m}\,t$$

solution oscillates without decay



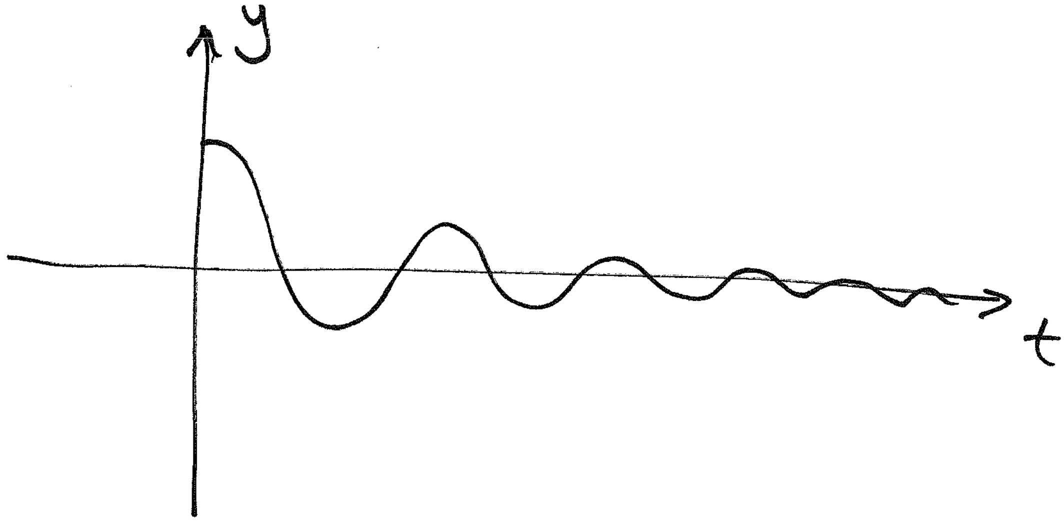
Case 2: $0 < b < \sqrt{4km}$ (underdamped)

$$\lambda_1 = -\frac{b}{2m} \pm i \frac{\sqrt{4mk - b^2}}{2m}$$

$$= -\alpha \pm i\beta$$

$$y = c_1 e^{-\alpha t}\cos\beta t + c_2 e^{-\alpha t}\sin\beta t$$

oscillates & decays



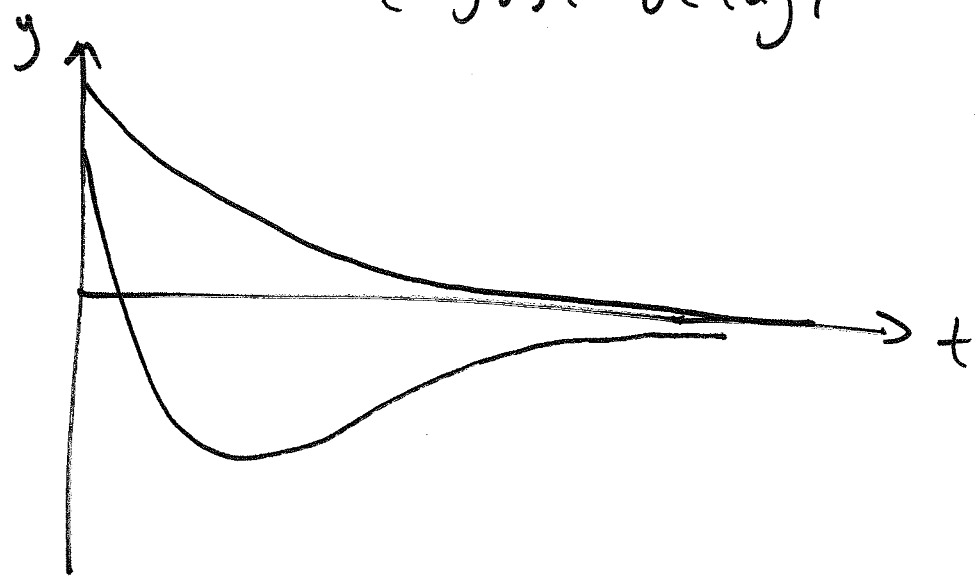
Case 3: $b > \sqrt{4km}$ (overdamped)

$$\lambda_1 = \frac{-b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m} < 0$$

$$\lambda_2 = \frac{-b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m} < 0$$

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

(just decay)

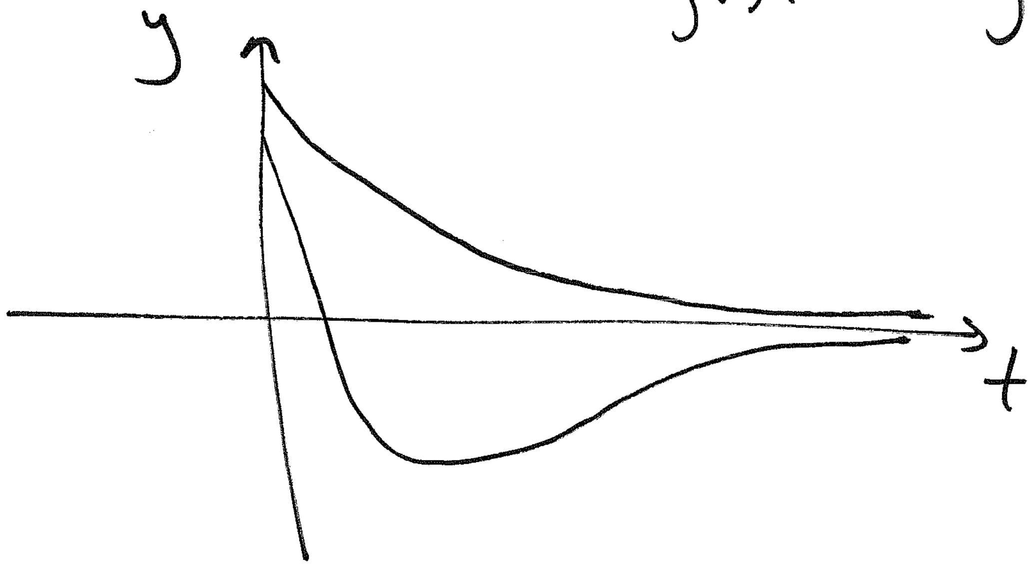


Case 4: $b = \sqrt{4km}$ (critical damping)

One root $\lambda_1 = \frac{-b}{2m}$

$$y = c_1 \cdot e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

just decay



Summary:

For the harmonic oscillator, modelled by the DE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

with constants $b \geq 0$ and $k > 0$:

- if $b = 0$ all solutions are periodic except the equilibrium at $x = 0$;
- if $b > 0$ all solutions tend to zero as $t \to \infty$.

Maths 260 Lecture 31

Topic for today

Nonhomogeneous higher order DEs

Reading for this lecture

BDH Section 4.1, 4.2

Suggested exercises

BDH Section 4.1; 1, 3, 7, 11,
Section 4.2; 1, 3, 9, 13

Reading for next lecture

BDH Section 4.3

Today's handout

Lecture 31 notes

Nonhomogeneous higher order linear DEs

An nth order linear DE of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

is called nonhomogeneous.

The function $f(t)$ is called the forcing function or nonhomogeneous term.

Example :

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = \sin t$$

models the behaviour of a mass/spring system subject to periodic forcing.

To solve a nonhomogeneous DE, we first solve the corresponding homogeneous equation and then combine this solution with a particular solution to the nonhomogeneous equation.

This result uses:

Extended Linearity Principle:

Given the nonhomogeneous DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

consider the corresponding homogeneous equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

1. If y_h is a solution to the homogeneous DE and y_p is a solution to the nonhomogeneous DE then $y_h + y_p$ is also a solution to the nonhomogeneous DE. has the unknown constants.

2. If y_c is the general solution to the homogeneous DE and y_p is a solution to the nonhomogeneous equation then $y = y_c + y_p$ is the general solution to the nonhomogeneous DE.

<u>Verification of extended linearity principle for</u>

$$y'' + py' + qy = f(t) \quad (*)$$

y_h satisfies

$$y_h'' + py_h' + qy_h = 0$$

y_p satisfies

$$y_p'' + py_p' + qy_p = f(t)$$

$y = y_h + y_p$

then $y'' + py' + qy$

$$= y_h'' + y_p'' + p(y_h' + y_p') + q(y_h + y_p)$$

$$= \underbrace{y_h'' + py_h' + qy_h}_{=0} + \underbrace{y_p'' + py_p' + qy_p}_{=f(t)}$$

$$= f(t)$$

i.e. $y = y_h + y_p$ satisfies $(*)$

Example: Show that $y_p = -2e^{-3t}$ is a solution to the equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2e^{-3t} \qquad (*)$$

Hence find the general solution to this equation.

$$y_p'' + 6y_p' + 8y_p = -18e^{-3t} + 36e^{-3t} - 16e^{-3t} = 2e^{-3t}$$

Now find y_h

$$y_h'' + 6y_h' + 8y_h = 0$$

solve $\lambda^2 + 6\lambda + 8 = 0$

$$\lambda = -4 \text{ or } -2$$

$$y_h = c_1 e^{-4t} + c_2 e^{-2t}$$

soln to $(*)$

$$y = y_p + y_h$$

$$= -2e^{-3t} + c_1 e^{-4t} + c_2 e^{-2t}$$

Summary of method of solution for nonhomogeneous equations

1. Find the general solution to the related homogeneous equation.
2. Find one solution to the nonhomogeneous equation.
3. Add answers to (1) and (2) to get the general solution to the nonhomogeneous equation.
4. If trying to solve an IVP, use the initial conditions to determine constants in the general solution.

Finding a particular solution

We saw (in computer demonstrations) that when the harmonic oscillator is subjected to external forcing, solutions frequently mimic the forcing, at least in the long term.

We use this observation as the basis of a method for finding particular solutions to linear, constant coefficient DEs.

Example 1 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^t$$

Find y_h

$$y_h'' + 3y_h' + 2y_h = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -2, -1$$

$$y_h = c_1 e^{-2t} + c_2 e^{-t}$$

Guess $y_p = Ae^t$

where we have to find A

$$y_p'' + 3y_p' + 2y_p = e^t$$

$$\Rightarrow \quad Ae^t + 3Ae^t + 2Ae^t = e^t$$

$$\Rightarrow \quad \cancel{8}\ 6Ae^t = e^t$$

$$A = 1/6$$

$$y_p = 1/6\, e^t$$

g.s. (general soln) $\quad y = y_h + y_p$

$$= c_1 e^{-2t} + c_2 e^{-t} + 1/6\, e^t$$

Example 2 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t$$

$$y_h = c_1 e^{-2t} + c_2 e^{-t}$$

guess $y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$$y_p'' + 3y_p' + 2y_p$$

$$= -A\cos t - B\sin t + 3(-A\sin t + B\cos t) + 2(A\cos t + B\sin t)$$

$$= (-A + 3B + 2A)\cos t \quad (-B - 3A + 2B)\sin t = \cos t$$

$\Rightarrow -A + 3B + 2A = 1$

$\Rightarrow -B - 3A + 2B = 0$

$$\left.\begin{aligned} A + 3B &= 1 \\ -3A + B &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= 1/10 \\ B &= 3/10 \end{aligned}$$

$$y_p = 1/10 \cos t + 3/10 \sin t$$

$$y = y_h + y_p$$

$$= c_1 e^{-2t} + c_2 e^{-t}$$

$$+ 1/10 \cos t + 3/10 \sin t$$

Example 3 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t^2$$

We Know ~~yp~~ y_h

guess $y_p = At^2 + Bt + C$

$y_p' = 2At + B$

$y_p'' = 2A$

$y_p'' + 3y_p' + 2y_p$

$= 2A + 3(\cancel{A}\,2At + B) + \cancel{8}$

$+ 2(At^2 + Bt + C)$

$= 2At^2 + (2B + 6A)t + 2C + 3B + 2A$

$= t^2$

$\Rightarrow 2A = 1, \quad 2B + 6A = 0, \quad 2C + 3B + 2A = 0$

$A = 1/2, \quad B = -3/2, \quad C = \frac{1}{2}(9/2 - 1)$

$= 7/4$?

$y_p = 1/2\, t^2 - 3/2\, t + 7/4$

$y = c_1 e^{-2t} + c_2 e^{-t} + 1/2\, t^2 - 3/2\, t + 7/4$

Example 4 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$$

guess $y_p = Ae^{-t}$

$$y_p'' + 3y_p' + 2y_p$$

$$= Ae^{-t} - 3Ae^{-t} + 2Ae^{-t}$$

$$= (A - 3A + 2A)e^{-t}$$

$$= 0$$

~~guess~~ $y_p = Ae^{-t}$ is a soln of homogeneous equation.

guess
$$y_p = Ate^{-t}$$
$$y_p' = Ae^{-t} - Ate^{-t}$$
$$y_p'' = Ate^{-t} - 2Ae^{-t}$$

substitute into equation

$$y_p'' + 3y_p' + 2y_p$$

$$= Ate^{-t} - 2Ae^{-t} + 3(Ae^{-t} - Ate^{-t}) + 2Ate^{-t}$$

$$= (A - 3A + 2A)te^{-t} + (-2A + 3A)e^{-t}$$

$$= e^{-t}$$

$\Rightarrow$ Since $A - 3A + 2A = 0$ we can solve

$$-2A + 3A = 1$$

$$A = 1$$

$$y_p = te^{-t}$$

$$y = c_1 e^{-t} + c_2 e^{-2t} + te^{-t}$$

We can formalise the guessing method used in the examples as:

Method of undetermined coefficients

To find a particular solution to the DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

where $f(t)$ is

(i) a constant, or

(ii) t^n for n a positive integer, or

(iii) $e^{\lambda t}$ for real $\lambda \neq 0$, or

(iv) $\sin bt$ or $\cos bt$, b constant, or

(v) a finite product of terms like (i)-(iv),

take the following steps:

1. Form the UC set consisting of f and all linearly independent functions obtained by repeated differentiation of f.