Maths 260 Lecture 21

Topics for today

Complex Numbers:

- Multiplication of polar forms
- De Moivre's formula
- Derivatives of complex-valued functions
- Euler's formula
- The exponential of a complex number

Reading for this lecture

The handout on complex numbers

BDH Appendix B

Suggested exercises

Problems at the back of the handout on complex numbers.

Reading for next lecture

BDH Section 3.4

Today's handouts

Lecture 21 notes

2.8.2 More on Complex Numbers

Multiplication of polar forms

Let

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

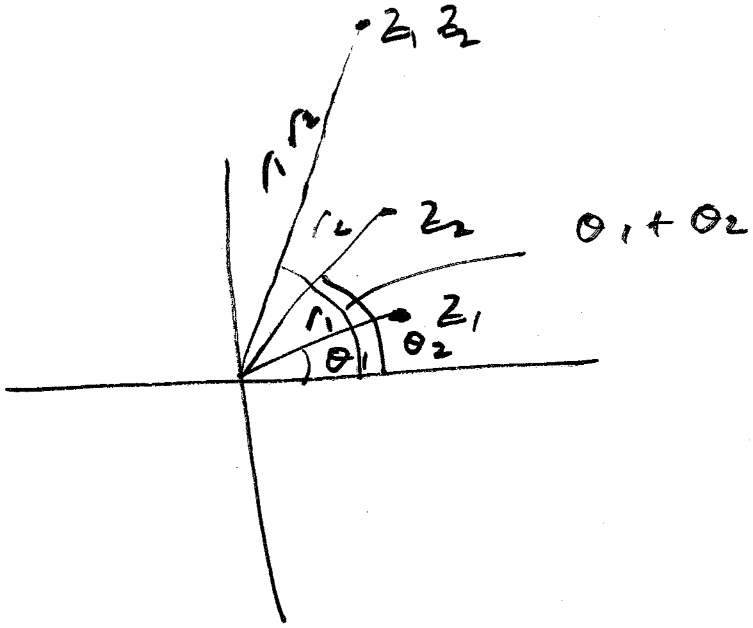
be any two complex numbers, then

$$
\begin{aligned}
z_1 z_2 &= \\
&= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))
\end{aligned}
$$

Hence, **multiplying** corresponds to

$$
\begin{aligned}
\textit{absolute value} &= \textit{product of absolute values} \\
\textit{argument} &= \textit{sum of arguments}
\end{aligned}
$$

Picture:



Example Solve $z^3 = 1$.

$z = 1$ is obviously a solution. Any others? Let's write

$$z = r(\cos\theta + i\sin\theta),$$

where $r = |z| > 0$. Then

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

and therefore

$$r^3(\cos 3\theta + i\sin 3\theta) = 1 = 1(\cos(0) + i\sin(0))$$

and

$\Rightarrow r^3 = 1 \quad (r = 1)$

$3\theta = 0 \Rightarrow \theta = 0$

but $0 = 2\pi = 4\pi$ etc as far as angle is concerned

Notice that for other values of n, the solutions given coincide with the above solutions because of the periodicity of cos and sin.

$\Rightarrow \quad 3\theta = 2\pi \Rightarrow \theta = 2\pi/3$

$3\theta = 4\pi \Rightarrow \theta = 4\pi/3$

$3\theta = 6\pi \Rightarrow \theta = 2\pi$

do not a new soln since $2\pi = 0$.

$z_1 z_1$

Example: Calculate

- $(\cos\theta + i\sin\theta)^2 =$ $\cos(\theta+\theta) + i\sin(\theta+\theta)$
- $(\cos\theta + i\sin\theta)^3 =$ $\cos(3\theta) + i\sin(3\theta)$

note $r_1 = 1$

These are particular cases of **de Moivre's formula:**

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta),$$

a very useful formula...

Example: Express $\cos 2\theta, \sin 2\theta$ in terms of $\cos\theta, \sin\theta$.

From the de Moivere's formula, we have

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

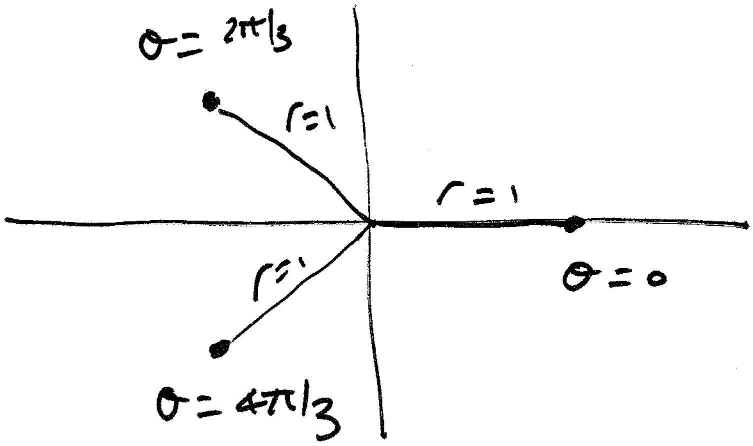
So $= \cos^2\theta + i\cos\theta\sin\theta + i\sin\theta\cos\theta - \sin^2\theta$

$\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow \sin 2\theta = 2\cos\theta\sin\theta$

Polar forms are sometimes useful for solving equations.

Plot the solutions: $\mathbb{C}$



Derivatives of complex valued functions

Suppose t is real and $f(t)$ is a complex valued function of t, i.e. t is real

$$f(t) = u(t) + iv(t) \qquad (u, v \text{ real functions})$$

Then, if u and v are differentiable at t, we define the derivative of $f(t)$ to be

$$\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}$$

Example Find the derivative of $f(t) = \cos(t) + i\sin(t)$

$$\left(f'(t) = -\sin t + i\cos t \right.$$
$$\left. (t \text{ is like } \theta) \right)$$

$(f = \cos t + i\sin t))$

Properties of $f(t)$:

- $f'(t) = if(t)$,
- $f(0) = 1$,
- $f(t_1)f(t_2) = f(t_1 + t_2)$. (from adding angles)

Compare this to $g(t) = e^{at}$, where a is real:

Properties of $g(t)$:

- $g'(t) =$, $a e^{at}$
- $g(0) =$, 1
- $g(t_1)g(t_2) =$. $e^{at_1} e^{at_2} = e^{a(t_1+t_2)} = g(t_1+t_2)$

Euler's Formula

The properties of f prompted Euler to make the definition:

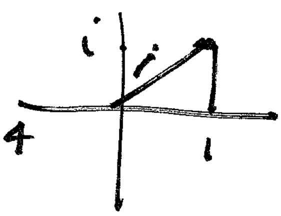
Euler's Formula: $\boxed{e^{it} = \cos t + i \sin t}$

VERY IMPORTANT

Euler's Formula and Polar forms

Example: $z = 1 + i.$ $\Rightarrow r\cos\theta + i r \sin\theta$

$r = \sqrt{2}, \ \theta = \pi/4$



$\Rightarrow z = \sqrt{2}(\cos \pi/4 + i \sin \pi/4) = \sqrt{2} e^{i\pi/4}$

In general, complex number $z = a + ib$ can be written in polar form as

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

Now multiplication and division are easy:

Example: $z_1 = 2e^{i\pi/6}$, $z_2 = e^{i\pi/4}$

$z_1 z_2 = 2 e^{i\pi(1/6 + 1/4)} = 2 e^{i\pi(5/12)}$

$z_1/z_2 = \frac{2}{1} e^{i\pi(1/6 - 1/4)} = 2 e^{i\pi(-1/12)}$

Also we can easily calculate **powers**:

Example 1: If $z = 3e^{i\pi/5}$, find z^2 and z^5.

$$z^2 = 3^2 e^{i\pi(1/5+1/5)} = 9e^{i\pi(2/5)}$$

$$z^5 = 3^5 e^{i\pi(1/5+1/5+1/5+1/5+1/5)}$$

$$= 243\, e^{i\pi} = -243$$

Example 2: Find all solutions of $z^3 = \cancel{8}.\ 2$

& $z^3 = 1+i$

$$z^3 = 2, \quad z = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta} = re^{i\theta}$$

$$\Rightarrow r^3 = 2, \quad 3\theta = 0, 2\pi, 4\pi$$

$$\Rightarrow r = \sqrt[3]{2}, \quad \theta = 0, 2\pi/3, 4\pi/3$$

$$z_1 = \sqrt[3]{2}\, e^{i0}, \quad z_2 = \sqrt[3]{2}\, e^{2\pi/3}, \quad z_3 = \sqrt[3]{2}\, e^{i4\pi/3}$$

$$\sqrt[3]{2}\, e^{i0} = 1$$

$$z^3 = 1+i = \sqrt{2}\, e^{i\pi/4}$$

$$\Rightarrow r = \sqrt[6]{\sqrt[3]{\sqrt{2}}} = \sqrt[6]{2}, \quad 3\theta = \pi/4,\ \pi/4+2\pi,\ \pi/4+4\pi$$

$$z_1 = \sqrt[6]{2}\, e^{i\pi/12}, \quad z_2 = \sqrt[6]{2}\, e^{i(\pi/12+2\pi/3)}, \quad z_3 = \sqrt[6]{2}\, e^{i(\pi/12+4\pi/3)}$$

The Exponential of a Complex Number

We know how to calculate e^x when x is real and e^{iy} when y is real, so it makes sense to define:

Definition: $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

Example: Calculate $e^{\log(2)+i\pi}$. $\cancel{= e}$

$$= e^{\log(2)} \cdot e^{i\pi}$$

$$= 2 \cdot -1 = -2 \left(= e^{\log(2)+i\pi}\right)$$

Example: Show that if λ is a complex number then

$$\frac{d}{dt}\left(e^{\lambda t}\right) = \lambda e^{\lambda t} \quad , \lambda = x + iy$$

$$\frac{d}{dt}\left(e^{\lambda t}\right) = \frac{d}{dt}\left(e^{(x+iy)t}\right)$$

$$= \frac{d}{dt}\left(e^{xt} e^{iyt}\right)$$

$$= x e^{xt} e^{iyt} + iy e^{xt} e^{iyt}$$

$$= (x+iy) e^{xt} e^{iyt}$$

$$= \lambda e^{\lambda t}$$

Example: Find all solutions of the form $y = e^{\lambda t}$ to the differential equation

$$y''(t) + 2y'(t) + 10y(t) = 0.$$

$$y = e^{\lambda t}, \quad y' = \lambda e^{\lambda t}, \quad y'' = \lambda^2 e^{\lambda t}$$

$$y'' + 2y' + 10y = \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 10 e^{\lambda t} = 0$$

$$\Rightarrow (\lambda^2 + 2\lambda + 10) e^{\lambda t} = 0$$

(true for all t)

$$\Rightarrow \lambda^2 + 2\lambda + 10 = 0 \quad (e^{\lambda t} \neq 0 \text{ for any } t)$$

soln $\quad$ soln $\lambda = -1 \pm 3i$

$$y = e^{(-1+3i)t} = e^{-t}(\cos 3t + i \sin 3t)$$

Note: We'll see later that the *general solution* of such equations can be found by taking a linear combination of the real and imaginary parts of the complex exponential solutions. So the general solution of the equation above is

$$y = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t.$$

soln to $y'' + 2y' + 10y = 0$

covered later in course

Maths 260 Lecture 22

Topic for today

Linear systems with complex eigenvalues

Reading for this lecture

BDH Section 3.4

Suggested exercises

BDH Section 3.4; 1, 3, 5, 7, 9, 11, 23

Reading for next lecture

BDH Section 3.5

Today's handouts

Lecture 22 notes

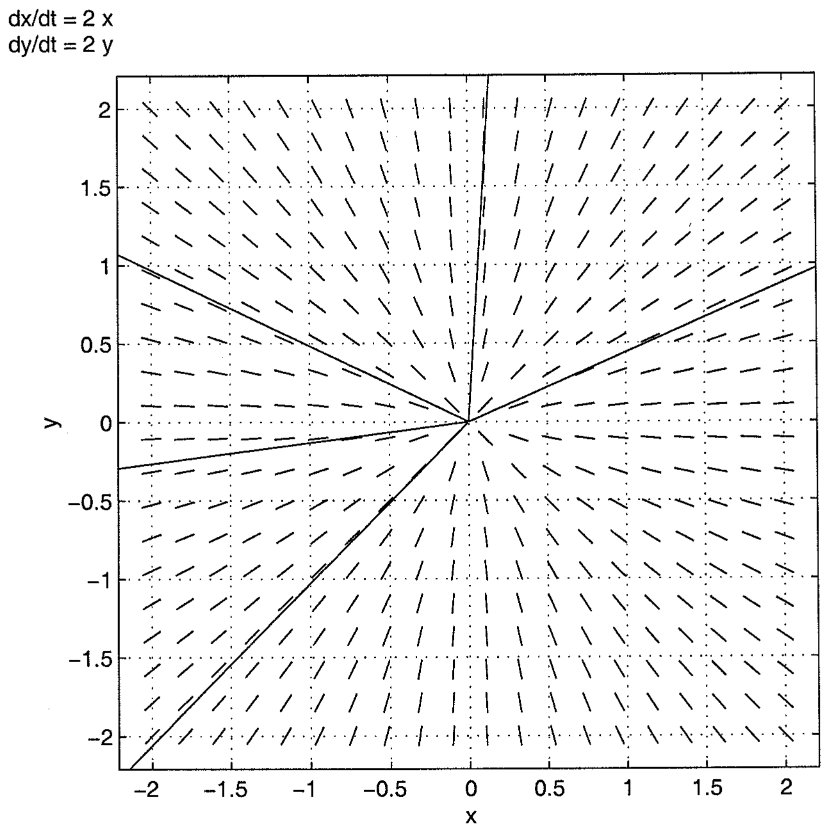
2.8.3 Linear systems with complex eigenvalues

There exist linear systems for which there are no straight-line solutions.

Example: Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

Slope field and some solutions



What goes wrong?

Calculate the eigenvalues:

$$\det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda + 1 + 4 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = 1 \pm 2i$$

See that eigenvalues are complex. We saw earlier that straight-line solutions result from real eigenvalues.

That is, $\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$ is a solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{v}$ but the corresponding solution curve will not be a straight-line if λ is not real.

Find (complex) solution vectors for this example:

$\lambda = 1+2i$

Solve $\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} \vec{v} = 0$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \vec{v}_1 = 0$$

$\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow -2i\alpha - 2\beta = 0 \quad (*)$

or $2\alpha - 2i\beta = 0 \quad (**)$

$(*)$ & $(**)$ are multiples of each other

$((*)\, i = (**))$

solve $(*)$, $\beta = 1 \Rightarrow \alpha = i$

$\vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

How do we interpret a complex-valued solution? We would like a real-valued solution.

Solve for $\lambda = 1 - 2i$

solve $\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} \vec{v}_2 = 0$

$\Rightarrow \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \vec{v}_2 = 0$

(two equations are same)

$$\vec{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Note $\vec{v}_2$ is complex conjugate of $\vec{v}_1$

Solve for $\lambda = 1-2i$

Solve $\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} \vec{v}_2 = 0$

$\Rightarrow \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \vec{v}_2 = 0$

(two equations are same)

$\vec{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

Note $\vec{v}_2$ is complex conjugate of $\vec{v}$

Theorem

Consider the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

If $\mathbf{Y}(t)$ is a complex-valued solution to the system, write

$$\mathbf{Y}(t) = \mathbf{Y_R}(t) + i\mathbf{Y_I}(t)$$

real

Then $\mathbf{Y_R}(t)$ and $\mathbf{Y_I}(t)$ are real-valued solutions to the system and are linearly independent.

Proof $\left(\frac{dY}{dt} = AY \right)$

$$\Rightarrow \frac{dY_R}{dt} + i\frac{dY_I}{dt} = AY_R + iAY_I$$

$$\Rightarrow \frac{dY_R}{dt} = AY_R, \quad \frac{dY_I}{dt} = AY_I$$

Apply theorem to previous example. Know that

$$V_1 = e^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

is a solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\mathbf{Y}.$$

But

$$e^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^t(\cos 2t + i\sin 2t)\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$= e^t\begin{pmatrix} i\cos 2t - \sin 2t \\ \cos 2t + i\sin 2t \end{pmatrix}$$

Hence, by theorem,

$$\mathbf{Y_R} = e^t\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$

and

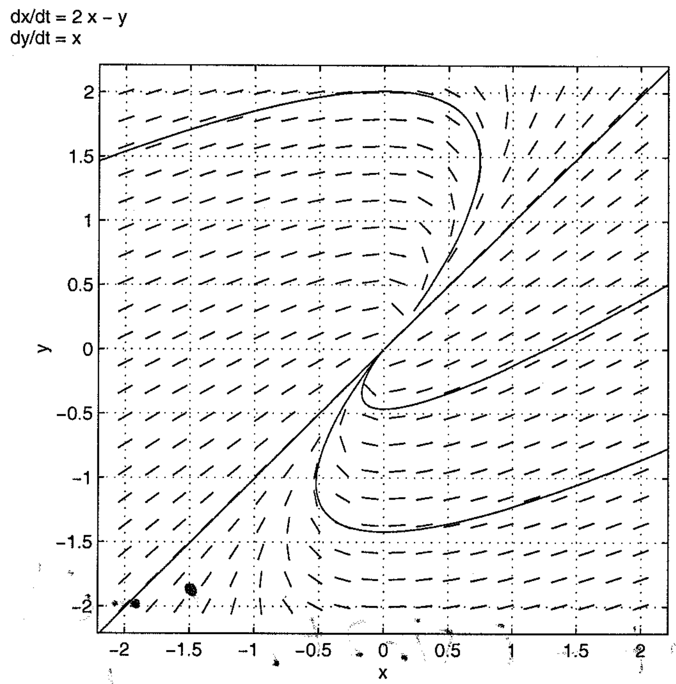
$$\mathbf{Y_I} = e^t\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

are real-valued, linearly independent solutions and the general solution is

$$Y = C_1 e^t\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix} + C_2 e^t\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

We see from the general solution that each component of $\mathbf{Y}$ will oscillate from positive to negative and that amplitude of each component will grow exponentially.

<u>Phase portrait</u>



<u>Components of solution with $x(0) = 1, y(0) = 0$</u>

$$Y(0) = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad c_1 = 0, \quad c_2 = 1$$

$$Y(t) = e^t \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$

Note : In this example, we found two linearly independent real-valued solutions by taking the real and imaginary parts of the complex-valued solution

$$e^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

What if we instead used the real and imaginary parts of the other complex-valued solution we found, i.e.,

$$e^{(1-2i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= e^{t}(\cos 2t - i\sin 2t)\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} -i\cos 2t - \sin 2t \\ \cos 2t - i\sin 2t \end{pmatrix}$$

$$Y_R = e^{t}\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}, \quad Y_I = e^{t}\begin{pmatrix} -\cos 2t \\ -\sin 2t \end{pmatrix}$$

We see that the other complex-valued solution also gives us two real-valued solutions but these solutions are just multiples of the real-valued solutions already found.

Thus, using the other complex-valued solution gives no new information; we can form the general solution using the real and imaginary parts of just one of the complex conjugate pair of solutions.

In general, the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

with complex eigenvalues

$$\lambda_1 = \alpha + i\beta$$

and

$$\lambda_2 = \alpha - i\beta$$

has a solution of the form

$$\mathbf{Y}(t) = e^{(\alpha+i\beta)t}\mathbf{Y}_0,$$

where $\mathbf{Y}_0$ is the eigenvector corresponding to eigenvalue $\lambda_1 = \alpha + i\beta$.

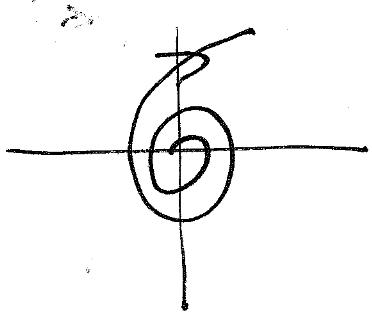
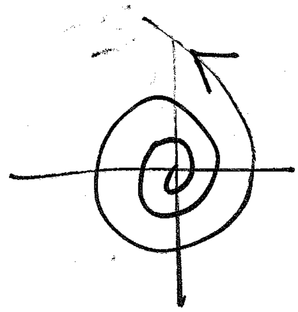
Expanding:

$$\mathbf{Y}(t) = e^{(\alpha+i\beta)t}\mathbf{Y}_0 = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\mathbf{Y}_0.$$

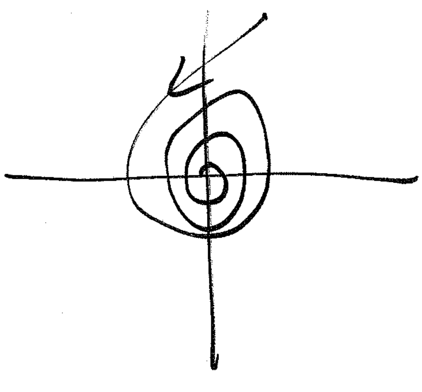
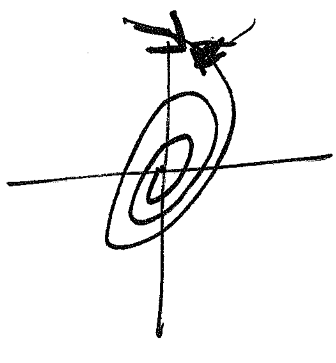
So the general solution is a combination of exponential and trigonometrical terms. The qualitative behavior of solutions depends on α and β.

When $\mathbf{A}$ is a 2×2 matrix, trig terms alternate between positive and negative with period $\frac{2\pi}{\beta}$, so the solution curves spiral around the origin in the phase plane.

1. If $\alpha > 0$, then $e^{\alpha t} \to \infty$ as $t \to \infty$ so solution curves spiral away from the origin. In this case, the equilibrium at the origin is called a spiral source.
   Typical phase portraits:

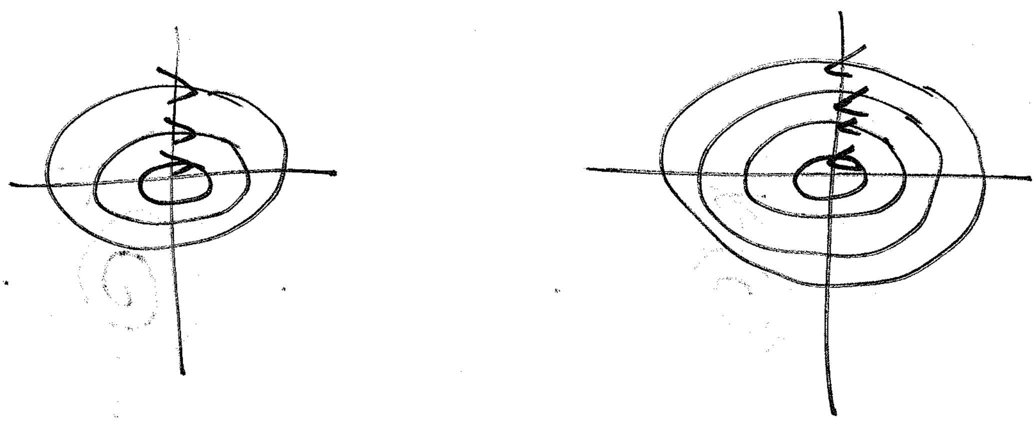



2. If $\alpha < 0$, then $e^{\alpha t} \to 0$ as $t \to \infty$ so solution curves spiral into the origin. In this case, the equilibrium at the origin is called a spiral sink.
   Typical phase portraits:

3. If $\alpha = 0$, then $e^{\alpha t} = 1$ and solution curves are periodic; solutions return to their initial conditions in the phase plane and repeat the same curve over and over again. In this case, the equilibrium at the origin is called a <u>centre</u>.

   Typical phase portraits:



Examples

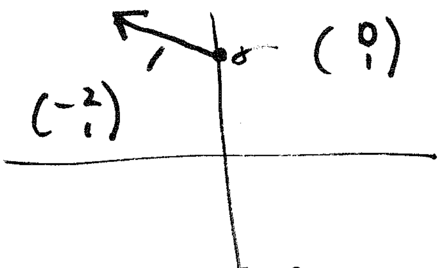
1. Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

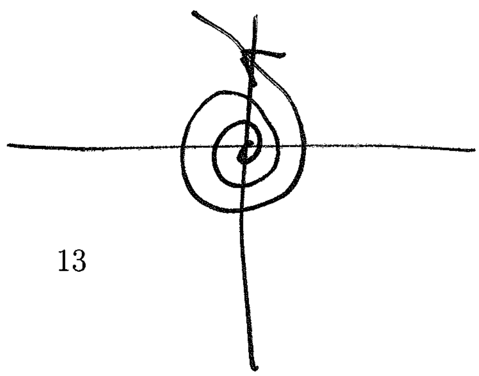
As before, e-values are $1 \pm 2i$ ($\alpha = 1$, $\beta = 2$) so origin is a spiral source.

To determine whether spiral is clockwise or anticlockwise, evaluate vector field at a point. For example, at $(x, y) = (0, 1)$ on the y-axis, vector field is

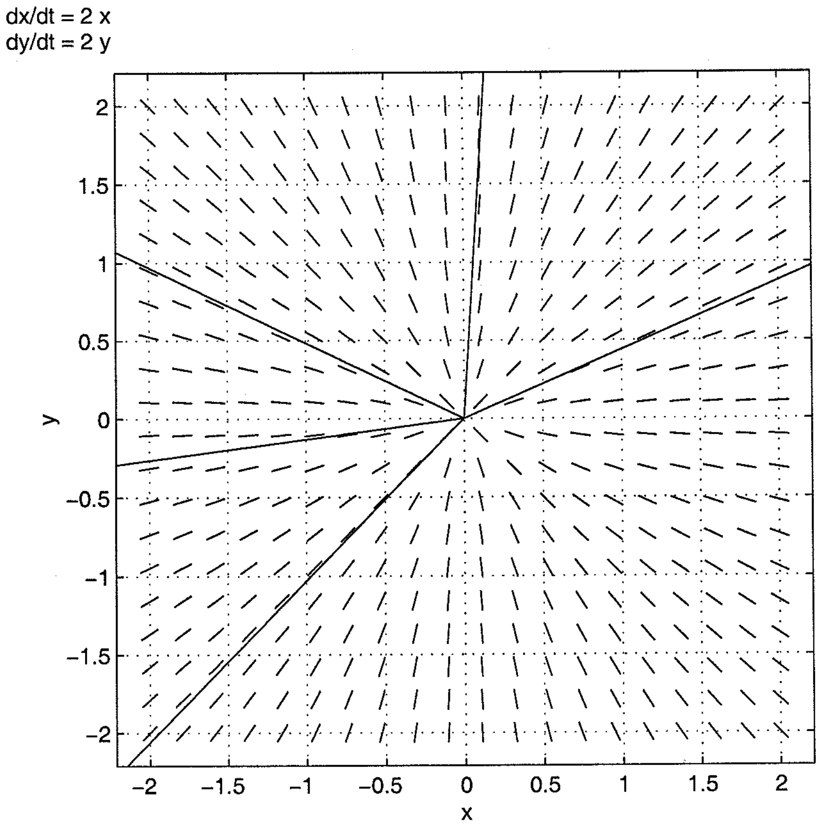
direction vector at $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\rightarrow$ $\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$



which points to left, so spiral is anticlockwise.



Direction field and some solutions



Exercise: Show that the general solution to the system, written in terms of real functions, is

$$Y(t) = c_1 e^t \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

2. Sketch the phase portrait for the system

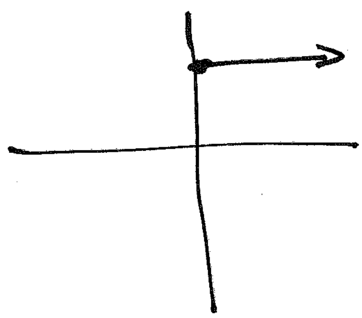
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

$$\det \begin{pmatrix} -2-\lambda & 3 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 3 = 0 \quad \text{use formula}$$

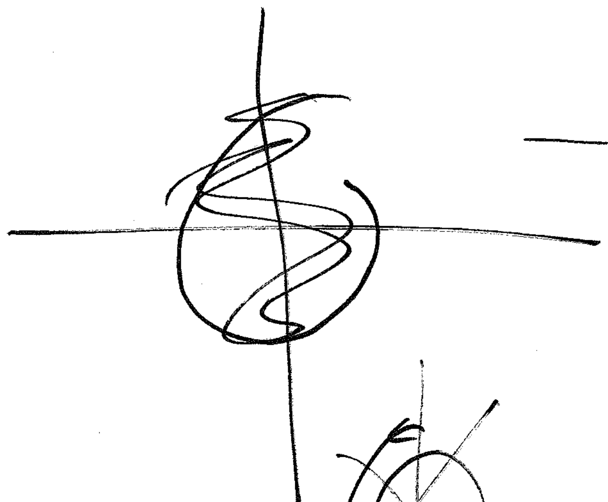
$$\lambda = -1 \pm \sqrt{2}\, i$$

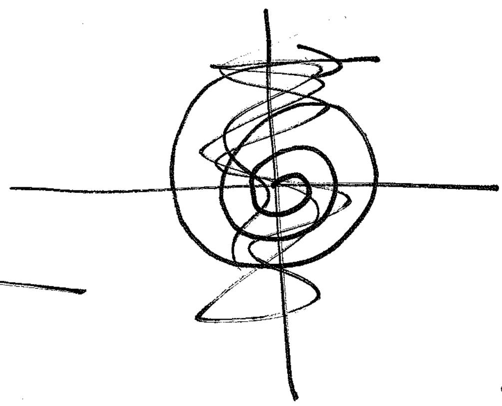
spiral sink

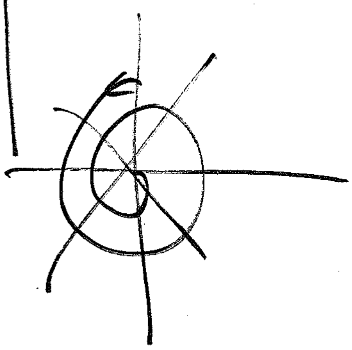
at $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

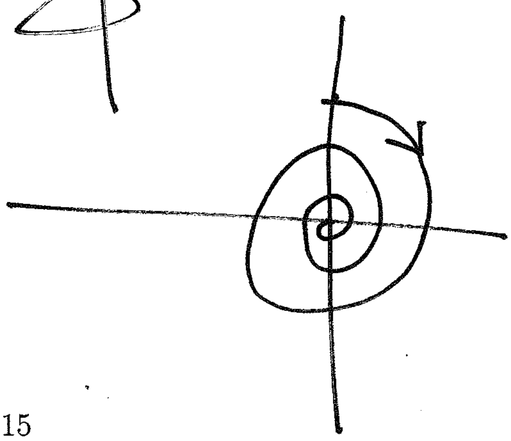


$\Rightarrow$ clockwise

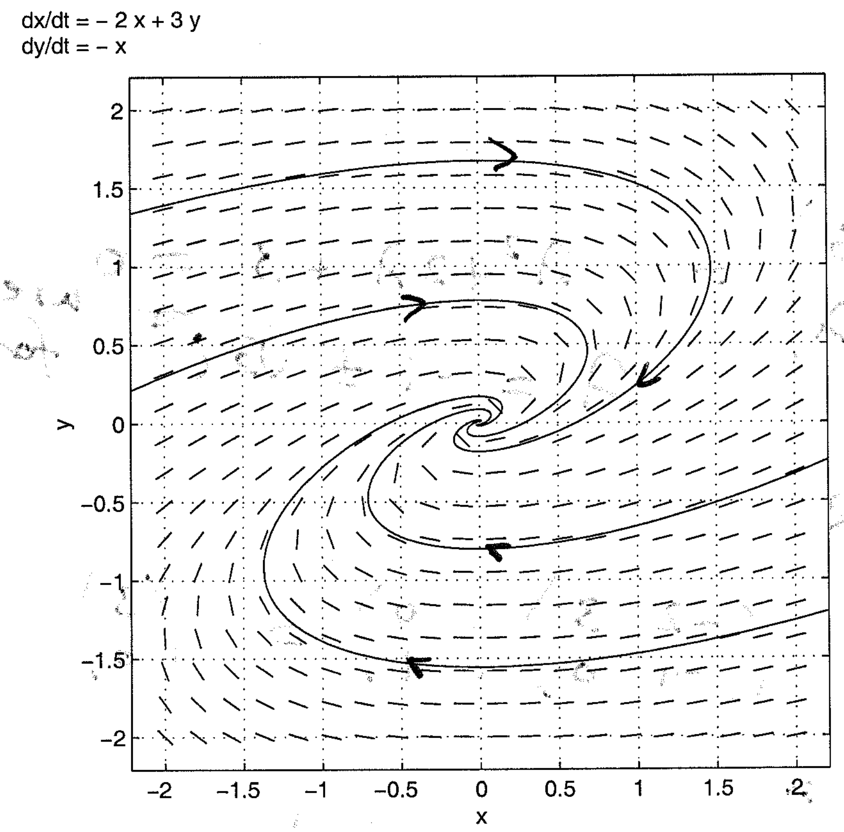








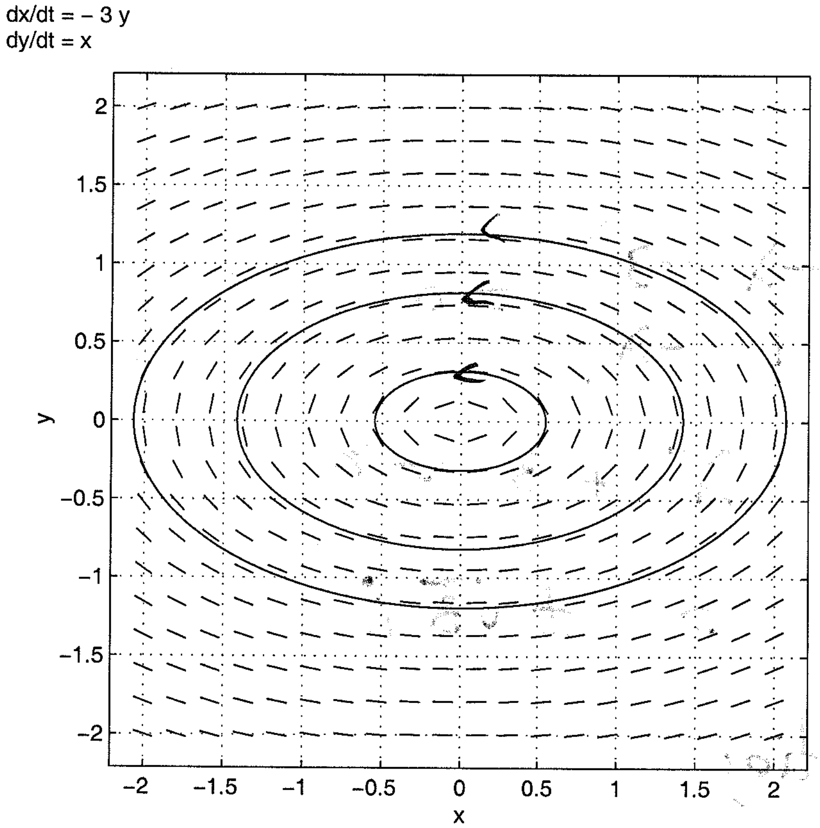
Direction field and some solutions



Exercise: Show that the general solution to the system, written in terms of real functions, is

$$\mathbf{Y}(t) = c_1\ e^{-t}\begin{pmatrix}\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \\ \cos(\sqrt{2}t)\end{pmatrix}$$
$$+c_2 e^{-t}\begin{pmatrix}\sin\sqrt{2}t - \sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t\end{pmatrix}$$

Direction field and some solutions



Exercise: Show that the general solution to the system, written in terms of real functions, is

$$\mathbf{Y}(t) = c_1 \begin{pmatrix} 3\cos\sqrt{3}t \\ \sqrt{3}\sin\sqrt{3}t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t \end{pmatrix}$$

3. Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

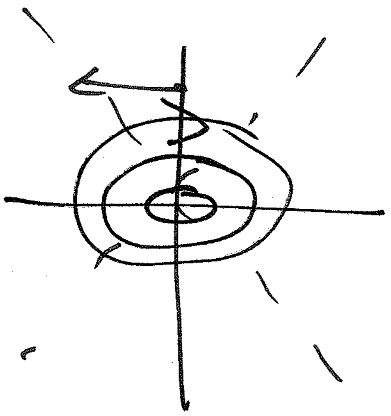
e-values

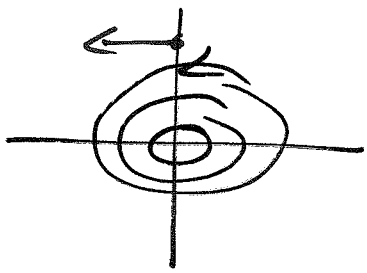
$$\det \begin{pmatrix} -\lambda & -3 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + 3 = 0$$

$$\lambda = \pm \sqrt{3}\, i$$

Two possibilities





check at $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

4. Find the general solution (expressed in terms of real functions) for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{pmatrix} \mathbf{Y}.$$

Determine the long term behaviour of solutions.

E-values are $1,\ 2+3i,\ 2-3i$ with corresponding e-vectors

$$\begin{pmatrix} -10 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

respectively.