Maths 260 Lecture 9

Topics for today
Classification of equilibria
Linearization

Reading for this lecture
BDH Section 1.6, pp 86-91

Suggested Exercises
BDH Section 1.6: 1, 3, 5, 7, 13, 15, 17

Reading for next lecture
BDH Section 1.7

Today's handouts
Lecture 9 notes
Assignment 2

Classifying Equilibria

To draw the phase line for a DE

$$\frac{dy}{dt} = f(y)$$

we need to know the positions of all equilibria, the intervals of y where $f(y) > 0$ and the intervals of y where $f(y) < 0$. If f is continuous, the sign of f can only change at y values where $f(y) = 0$, i.e., at equilibria. Thus, the positions of the equilibria and the behaviour of solutions near each equilibrium is all we need to draw the phase line.

Example

We classify equilibria according to the behaviour of nearby solutions.

1. An equilibrium $y = a$ is a sink if any solution with initial condition sufficiently close to a tends to a as t increases.

2. An equilibrium $y = b$ is a source if any solution with initial condition sufficiently close to b tends away from b as t increases (which means nearby solutions diverge from b as t increases.)

3. An equilibrium that is neither a sink nor a source is called a node.

Example 1.

$$\frac{dy}{dt} = y(3 + y)$$

Example 2.

$$\frac{dy}{dt} = y(y + 2)^2$$

Example 3.

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ has graph shown.

Linearization

If y_0 is an equilibrium solution of $\frac{dy}{dt} = f(y)$ and is a sink, then the phase line near y_0 looks like

which means

- $f(y) > 0$ if $y < y_0$
- $f(y) < 0$ if $y > y_0$
- $f(y_0) = 0$

So $f(y)$ is a decreasing function near y_0.

If y_0 is an equilibrium solution of $\frac{dy}{dt} = f(y)$ and is a source, then the phase line near y_0 looks like

which means

- $f(y) < 0$ if $y < y_0$
- $f(y) > 0$ if $y > y_0$
- $f(y_0) = 0$

So $f(y)$ is a increasing function near y_0.

These examples motivate the following theorem:

Linearization Theorem Suppose that $y = y_0$ is an equilibrium point of the DE

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ and $\partial f/\partial y$ are both continuous.

1. If $f'(y_0) < 0$, then y_0 is a sink.
2. If $f'(y_0) > 0$, then y_0 is a source.
3. If $f'(y_0) = 0$, or if $f'(y_0)$ does not exist, then we need additional information to determine the type of y_0.

Note: In the last case, the equilibrium may be a node or a sink or a source.

Example 1:

For the DE

$$\frac{dy}{dt} = y^2(y-2)(y+2)$$

find all equilibrium solutions and classify them using the linearization theorem.

Example 2:

Consider the following population model:

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{200}\right)\left(\frac{P}{50} - 1\right)$$

Classify the equilibria, draw the phase line and sketch some solutions for P.

Important ideas from today

Equilibria are classified as sink, source or node depending on the behaviour of nearby solutions.

Linearization - we can sometimes use df/dy to classify equilibria.