Maths 260 Lecture 8

Topic for today
The phase line

Reading for this lecture
BDH Section 1.6, pp 76-85

Suggested Exercises
BDH Section 1.6: 23, 25, 27, 29

Reading for next lecture
BDH Section 1.6, pp 81-88

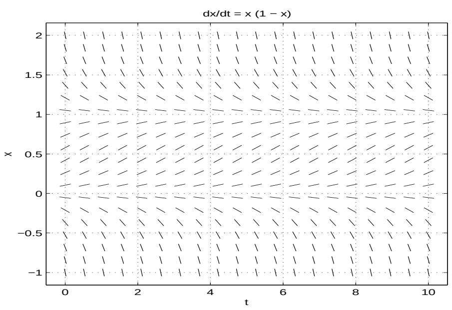
Today's handouts
Lecture 8 notes

§1.6 The Phase line

Consider the DE

$$\frac{dy}{dt} = f(y).$$

Recall that the slope field corresponding to an autonomous differential equation has a special form - slope marks are parallel along horizontal lines (see Lecture 4).



There is clearly some redundancy in slope field information. We can replace the slope field by a phase line, which summarises the information in the slope field.

To draw a phase line for $\frac{dy}{dt} = f(y)$

1. Draw y-line.
2. Find equilibrium solutions of the DE and mark them on the line.
3. Find intervals of y for which $f(y) > 0$ (solutions started at such y values will increase as t increases). Draw upward pointing arrows on the line in these intervals.
4. Find intervals of y for which $f(y) < 0$ (solutions started at such y values will decrease as t increases). Draw downward pointing arrows on the line in these intervals.

Examples

1. For the DE

$$\frac{dy}{dt} = (y+2)(1-y)$$

sketch the phase line. Describe the longterm behaviour of solutions.

2. For the DE

$$\frac{dy}{dt} = y^2(y+1)$$

sketch the phase line. Describe the longterm behaviour of solutions.

3. For the DE

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ has the graph shown below, sketch the phase line and describe the longterm behaviour of solutions.

It is possible to sketch solutions to a DE just from the phase line.

Longterm behaviour of solutions

In cases where the Uniqueness Theorem applies, a solution that tends to an equilibrium point does not reach the equilibrium point in finite time. We write

$$y(t) \to y_0 \text{ as } t \to \infty \text{ (or as } t \to -\infty).$$

In contrast, a solution that tends to $+\infty$ or $-\infty$ may reach $\pm\infty$ in finite time or may never reach $\pm\infty$. We cannot tell which case we have from the phase line alone.

Example:

$$\frac{dy}{dt} = 1$$

Example:

$$\frac{dy}{dt} = 1 + y^2$$

These examples show that we cannot write

$$y(t) \to \pm\infty \quad \text{as} \quad t \to \infty \text{ (or as } t \to -\infty)$$

based on evidence from the phase line alone - we would need more information about the actual solutions before making such a statement. Instead, based on phase lines, we make statements like

$$y(t) \to \infty \quad \text{as} \quad t \text{ increases}$$

or

$$y(t) \to \infty \quad \text{as} \quad t \text{ decreases.}$$

Main ideas for today

For an autonomous differential equation $\frac{dy}{dt} = f(y)$ it can be useful to sketch the phase line. The phase line contains information about equilibrium solutions and whether other solutions are increasing or decreasing, but information about the speed with which solutions are changing is lost.