Maths 260 Lecture 7

Topic for today
Existence and uniqueness of solutions

Reading for this lecture
BDH Section 1.5

Suggested Exercises
BDH Section 1.5: 1, 3, 5, 7, 15

Reading for next lecture
BDH Section 1.6, pp 74-80

Today's handouts
Lecture 7 notes

§1.5 Existence and Uniqueness of solutions

In the theory and examples we have studied already we have been making two major assumptions: that the DEs we study really have solutions and that such solutions are unique. On the whole we are safe in making these assumptions. Today we shall see why.

Existence Theorem
Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

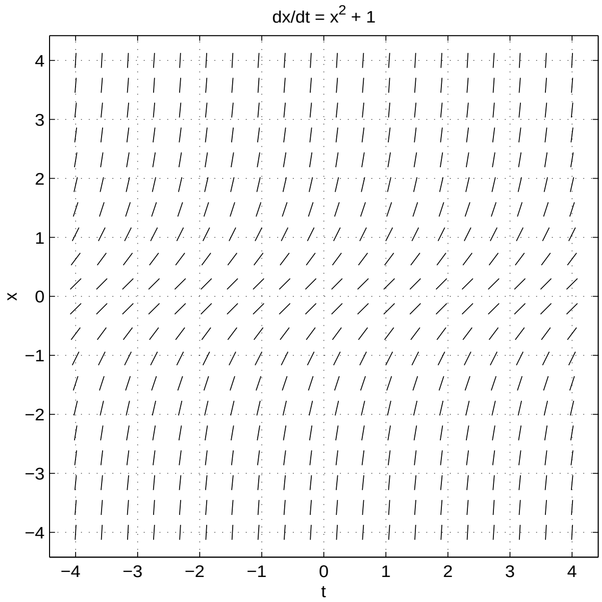
If $f(t, y)$ is a continuous function of t and of y at $(t, y) = (t_0, y_0)$, then there is a constant $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ solves the IVP.

Note: The theorem guarantees a solution exists for a small interval in t, but says nothing about existence for all t.

Example: Consider the IVP

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

Does the IVP have a solution? For what values of t does the solution exist? The slope field for the DE is shown on the next page.



Here $f(t, y) = 1 + y^2$ is a continuous function of t and of y for all t, y, so the Existence Theorem ensures a solution to the IVP exists for $-\epsilon < t < \epsilon$, for some ϵ.

In fact, $y(t) = \tan(t)$ is a solution to IVP and is defined for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ but not for all t.

Uniqueness Theorem

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

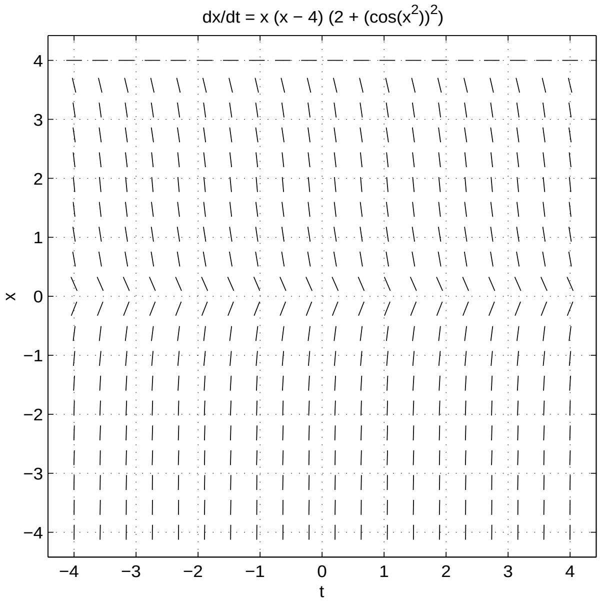
If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions of t and of y at $(t, y) = (t_0, y_0)$, then there is an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $y(t)$ is the unique solution to the IVP on this interval.

Note: The Uniqueness Theorem implies that different solutions can never cross or meet in (t, y) plane.

Examples where the Existence and Uniqueness Theorems are useful

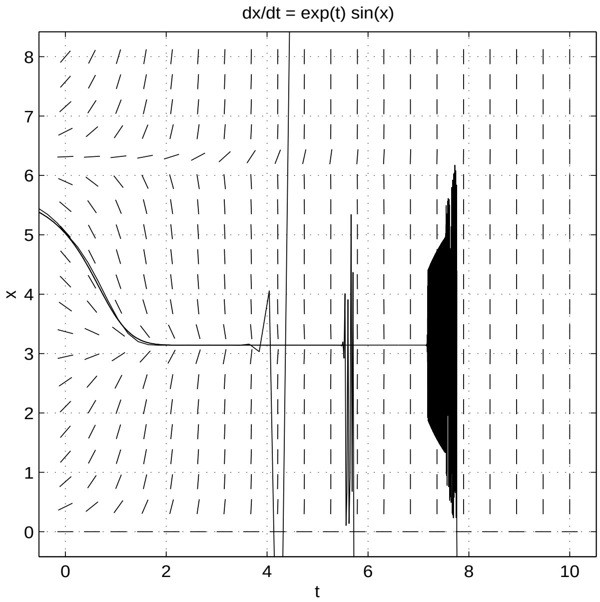
1. $\dfrac{dy}{dt} = y(y-4)(2+cos^2(y^2)), \quad y(0) = 1$

   What is the qualitative behaviour of solutions to the IVP?



2. For the IVP $\dfrac{dy}{dt} = e^t \sin(y), \quad y(0) = 5$ use the function *dfield* from Matlab and Euler's method with various step sizes to determine the behaviour of the solution to the DE.

   Plotting the solution with three different step sizes ($h = 0.02$, $h = 0.002$ and $h = 0.0002$) gives three qualitatively different approximate solutions:



3. Given the IVP

$$\frac{dy}{dt} = ty^{\frac{1}{5}}, \quad y(t_0) = y_0$$

(a) Find a value of t_0 and a value of y_0 so that the IVP has a unique solution. Give a reason for your answer.

(b) Find a value of t_0 and a value of y_0 so that the IVP has more than one solution. For your choice of t_0 and y_0 write down two functions that satisfy the DE.

Important ideas from today

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

If f is a continuous function of t and y at (t_0, y_0), a solution to the IVP exists, at least for t near t_0. If in addition $\frac{\partial f}{\partial y}$ is a continuous function of t and y at (t_0, y_0), the solution to the IVP is unique. This result implies that solution curves won't cross or touch at (t_0, y_0) when plotted in the $t - y$ plane.