Maths 260 Lecture 6

Topics for today
Order of numerical methods
Efficiency of numerical methods
Comparison of methods seen so far

Reading for this lecture
BDH Section 7.4

Suggested exercise
BDH Section 7.3: 6

Reading for next lecture
BDH Section 2.1

Today's handouts
Lecture 6 notes
Tutorial 2 Questions

Order of Numerical Methods

The **order** of a numerical method measures the change in error of a numerical solution as step size is decreased.

With Euler's method we find that the error is approximately halved if the step size is halved (at least if h is sufficiently small). For example, Euler's method gives us the following results for the IVP

$$\frac{dy}{dt} = y, \; y(0) = 1.$$

No. of Steps	Est of $y(1)$	error
1	2.000000	0.718
2	2.250000	0.468
4	2.441406	0.277
8	2.565784	0.152
16	2.637928	0.0804
32	2.676990	0.0413
64	2.687345	0.0209
128	2.707739	0.0105
256	2.712992	0.00529
512	2.715632	0.00265

Looking at the same IVP with Improved Euler yields:

No. of Steps	Est of $y(1)$	error
1	2.500000	0.218
2	2.640625	0.0777
4	2.694856	0.0234
8	2.711841	0.00644
16	2.716594	0.00169
32	2.717850	0.000432
64	2.718173	0.000109
128	2.718254	0.0000274
256	2.718275	0.00000689
512	2.718280	0.00000173

The same IVP with RK4 yields:

No. of Steps	Est of $y(1)$	error
1	2.708333	0.00994
2	2.717346	0.000936
4	2.718210	0.0000719
8	2.718277	0.00000498
16	2.718282	0.000000328
32	2.718282	0.0000000215

We see that:

1. for a fixed step size we get a smaller error with IE and RK4 than with Euler's method, i.e., IE and RK4 are more accurate methods than Euler's method for this problem;

2. (more importantly) for IE and RK4 the error decreases faster as the step size is reduced than with Euler's method.

Define $E(h)$ to be the error in the approximate solution obtained when we solve an IVP using a particular numerical method with step size h. If $|E(h)| \approx kh^p$ as $h \to 0$ then p is called the **order of the numerical method**. Here, k is a constant depending on the IVP and the method.

Estimating the order of a numerical method

For a particular IVP and a particular method of order p,

$$\lim_{h\to 0}\frac{|E(2h)|}{|E(h)|}\approx\lim_{h\to 0}\frac{k(2h)^p}{kh^p}=2^p$$

Taking logs,

$$\ln\left|\frac{E(2h)}{E(h)}\right|\approx p\ln 2$$

or, for small h,

$$p\approx\frac{\ln|E(2h)|-\ln|E(h)|}{\ln 2}$$

For a particular choice of h, the quantity

$$q=\frac{\ln|E(2h)|-\ln|E(h)|}{\ln 2}$$

is called the **effective order of the method** at step size h. We expect $q\to p$ as $h\to 0$.

Example: Consider the IVP

$$\frac{dy}{dt}=y,\quad y(0)=1$$

Estimates for $y(1)$ calculated with various step sizes were given earlier. Use these estimates to calculate the effective order of Euler's method for this IVP at $h=0.125$. Repeat for IE and RK4.

It can be proved that

- Euler's method is of order 1
- Improved Euler is of order 2
- Runge-Kutta 4 is of order 4

Efficiency of numerical methods

Higher order methods take more work per step than Euler's method. However, they usually require fewer steps and less total work to obtain an accurate answer. The most efficient method for a particular problem is the one which gives the desired accuracy with the least amount of work.

In estimating the amount of work required to use a certain method, we only count the number of evaluations of f. In comparison, additions and multiplications required are negligible.

Example: For the IVP discussed at the start of lecture which of Euler, IE and RK4 is most efficient if an accuracy of 1% in the solution at $t = 1$ is required?

The Dormand-Prince method

The Matlab functions *dfield* and *pplane* use the Dormand-Prince numerical method. We do not study this method in detail, but note that it incorporates three improvements over RK4:

1. It is a 5th order method.

2. A variable step size is used. The algorithm calculates the step size to be used by estimating the error in each step. A large error estimate will cause a smaller step size to be used.

3. Some fitting (with splines) is performed to give smoother solutions.

Comparison of Numerical Methods

1. **Euler's method**
   - order 1 - error reduces by a factor of approximately 2 as step size is doubled
   - 1 function evaluation per step.

2. **Improved Euler's method** - order 2 - error reduces by a factor of approximately $2^2 = 4$ as step size is doubled - 2 function evaluation2 per step.

3. **4th order Runge-Kutta method** - order 4 - error reduces by a factor of approximately $2^4 = 16$ as step size is doubled - 4 function evaluations per step.

4. **Dormand-Prince method** - order 5 - error reduces by a factor of approximately $2^5 = 32$ as step size is doubled. Variable stepsize method incorporating an error estimator. About 6 function evaluations per step.

Comparison of Methods seen in the course so far

We have seen examples of three important types of methods for getting information about solutions to DEs.

1. **Qualitative methods** (e.g. slope fields) are useful for understanding qualitative properties of solutions (e.g. long term behaviour) but do not give exact values of solutions at particular times.

2. **Analytic methods** (e.g. solving separable equations) give a formula for a solution of a DE. These methods work in some important special cases but do not work in most cases.

3. **Numerical methods** (e.g. Euler's method) give approximate quantitative information about solutions. We can automate these methods (i.e. use a computer). These methods can be misleading, and give information about only one solution at a time.

It is usually possible to use more than one method for any problem - the trick is picking the most appropriate method(s). We will learn more about each class of methods in the rest of the course.