Maths 260 Lecture 4

Topics for today
Slope fields
Euler's method

Reading for this lecture
BDH Sections 1.3, 1.4

Suggested Exercises
BDH Section 1.3: 11, 13, 15, Section 1.4: 7

Reading for next lecture
BDH Section 1.4 (again)

Section 1.3 Qualitative technique: Slope Fields

Slope fields provide a geometric technique for visualising the graph of a solution to

$$\frac{dy}{dt} = f(t, y)$$

without needing to first find a formula for $y(t)$. Assume $y(t)$ is a solution to

$$\frac{dy}{dt} = f(t, y).$$

Then at $t = t_1$, $y(t_1) = y_1$, and

$$\frac{dy}{dt} = f(t_1, y_1)$$

i.e., the slope of the graph of y at t_1 is $f(t_1, y_1)$.

Similar results hold for all other values of t, i.e., the slope of the graph of a solution $y(t)$ at $t = \bar{t}$ with $y(\bar{t}) = \bar{y}$ is given by $f(\bar{t}, \bar{y})$. We can use this result to draw a *slope field* which helps us sketch solutions to the DE.

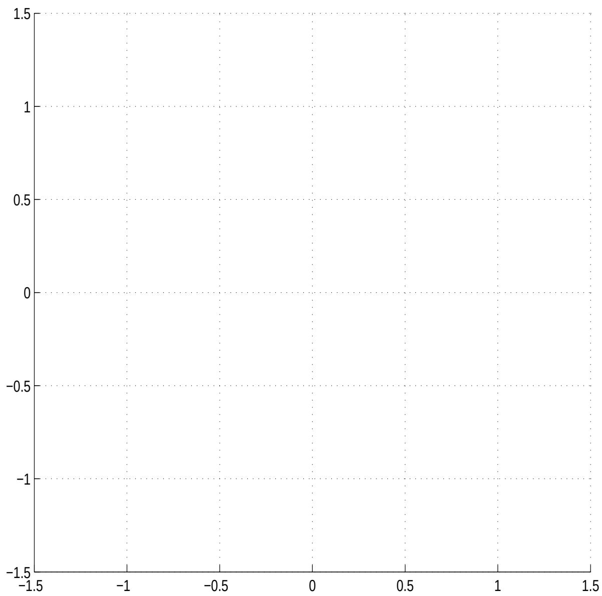
To draw a slope field:

1. For selected points in the $t - y$ plane (say at all points on an evenly spaced grid) calculate $f(t, y)$.
2. For each point $(\bar{t}, \bar{y})$ selected in (1), draw a short line segment of slope $f(\bar{t}, \bar{y})$ centered at $(\bar{t}, \bar{y})$.

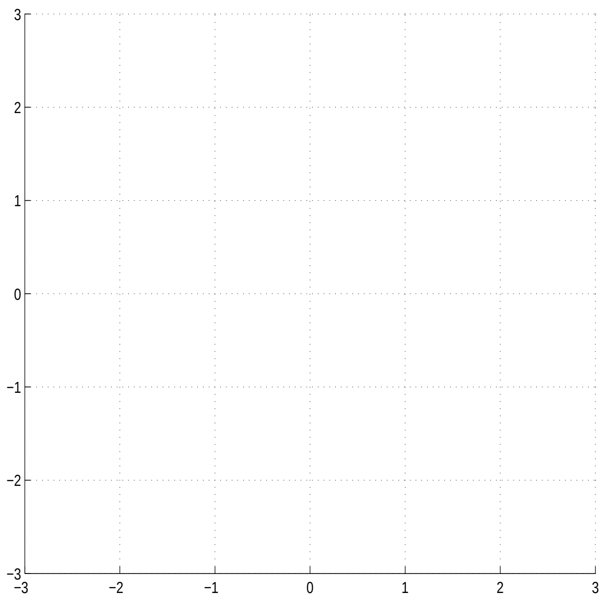
The resulting picture is called a slope field for the DE.

Examples

1. Use the following grid to draw the slope field for the DE $\dfrac{dy}{dt} = y - t$



2. Use the following grid to draw the slope field for the DE $\dfrac{dy}{dt} = -yt$

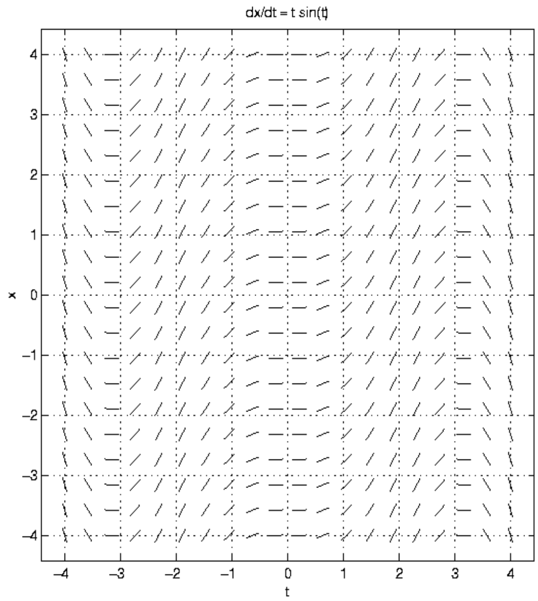


Sketching solutions using the slope field:

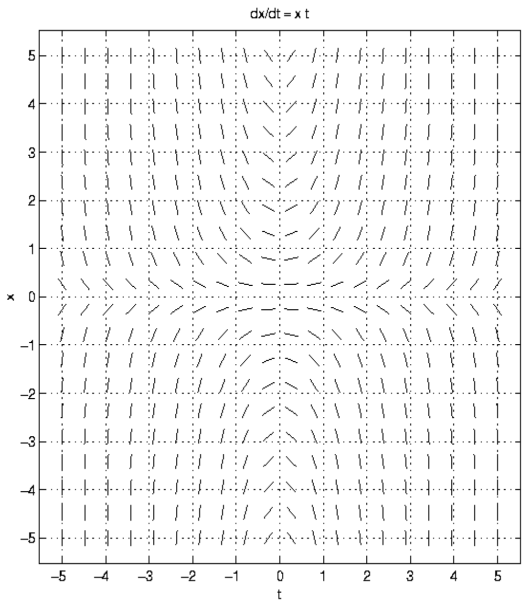
To sketch a solution to an IVP $\dfrac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$

1. Sketch the slope field as above.
2. Starting at the point (t_0, y_0) draw a curve that follows the direction field.

Example 1: The following picture shows the slope field for the DE $\dfrac{dy}{dt} = t \sin t$
Draw solutions to this DE satisfying initial conditions (a) $y(1) = 0$,(b) $y(0) = -1$.



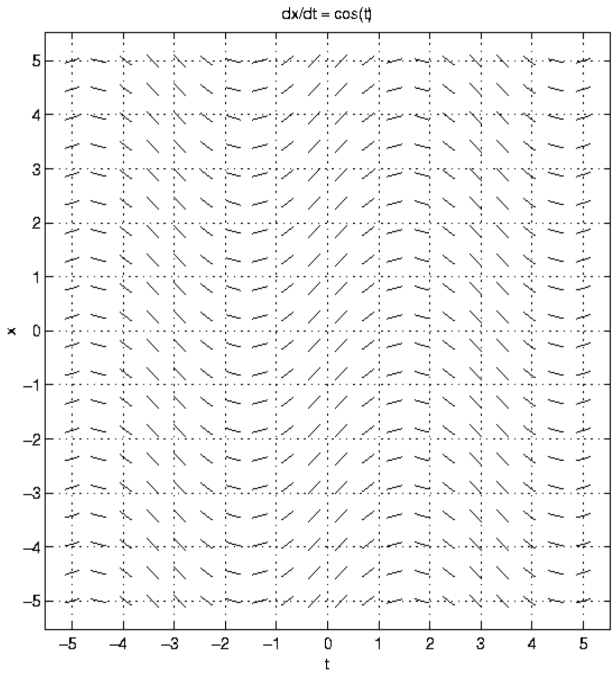
Example 2: The following picture shows the slope field for the DE $\dfrac{dy}{dt} = yt$.
Draw solutions to this DE satisfying initial conditions (a) $y(1) = 0$ and (b) $y(0) = -1$.



Two special cases

1. For differential equations of the form $\dfrac{dy}{dt} = f(t)$ all slope marks on each line of fixed t in the slope field are parallel.

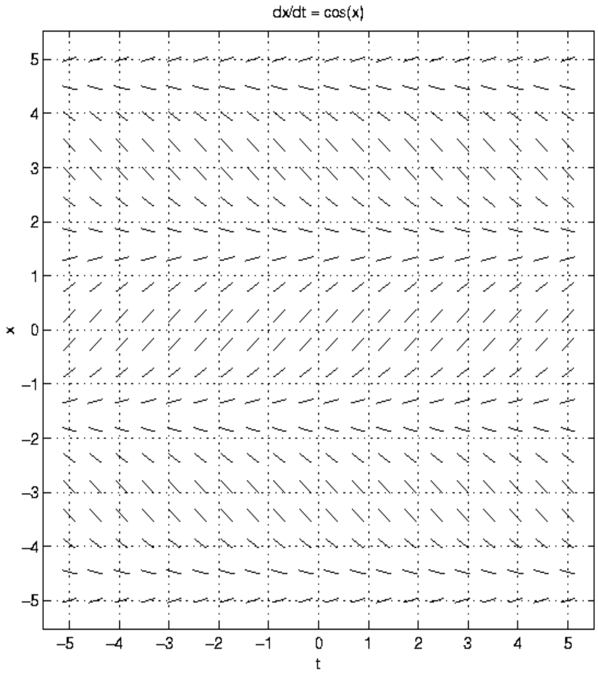
   Example: $\dfrac{dy}{dt} = cos(t)$.



   Note that if we have the graph of one solution, we can get graphs of other solutions by translating the graph vertically.

2. For differential equations of the form $\dfrac{dy}{dt} = f(y)$ all slope marks on each line of fixed x in the slope are parallel.

   Example: $\dfrac{dy}{dt} = cos(y)$.



   If we have the graph of one solution we can get graphs of other solutions by translating the graph horizontally.

<u>Section 1.4 Euler's method</u>

We can obtain numbers and graphs that approximate solutions to initial value problems using a class of techniques called numerical methods. Euler's method is the simplest numerical method, and is closely related to the slope field.

<u>Main idea:</u> For the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

start at (t_0, y_0) and take small steps, with the direction of each step being the direction of the slope field at the start of that step.

More formally, given (t_0, y_0) and a stepsize, h, we want to calculate an approximation to $y(t_1)$, $y(t_2)$, $y(t_3)$, etc where

$$\begin{aligned} t_1 &= t_0 + h, \\ t_2 &= t_1 + h = t_0 + 2h, \end{aligned}$$

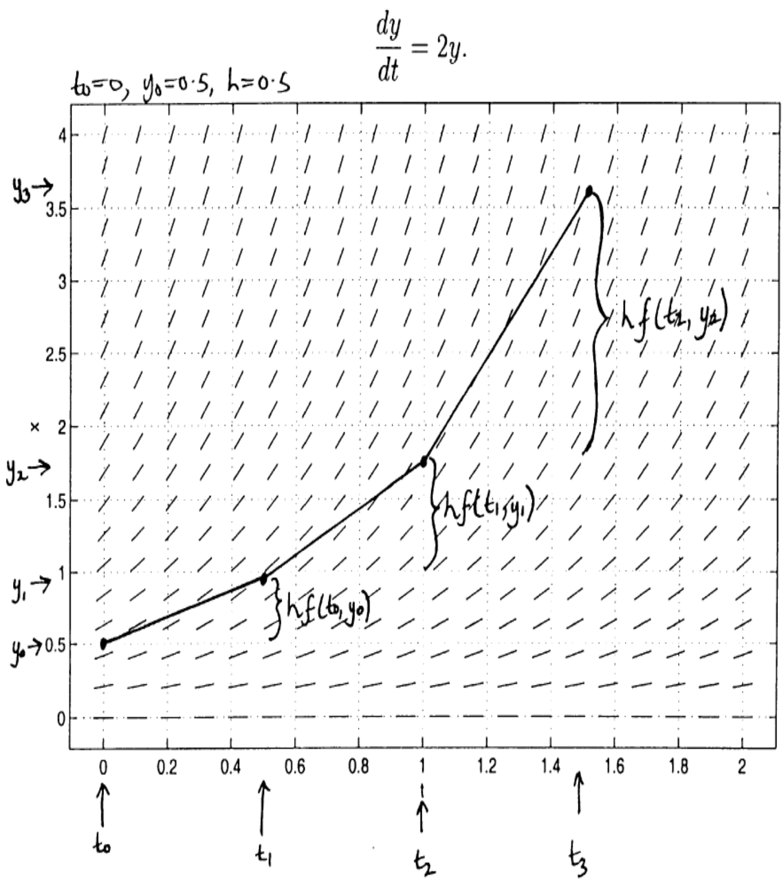
and so on. Slope of the direction field at (t_0, y_0) is $f(t_0, y_0)$ so

$$\begin{aligned} y(t_1) &\approx y_1 = y_0 + hf(t_0, y_0), \\ y(t_2) &\approx y_2 = y_1 + hf(t_1, y_1), \\ &\vdots \\ y(t_{k+1}) &\approx y_{k+1} = y_k + hf(t_k, y_k) \end{aligned}$$

for $k = 0, 1, ...n$.

The following picture illustrates the relationship between the slope field and the numerical solution obtained from Euler's method for the DE

$$\frac{dy}{dt} = 2y.$$



Example 1 Use Euler's method to approximate the solution of the IVP

$$\frac{dy}{dt} = \sqrt{t^2 + y^2}, \quad y(0) = 0.75$$

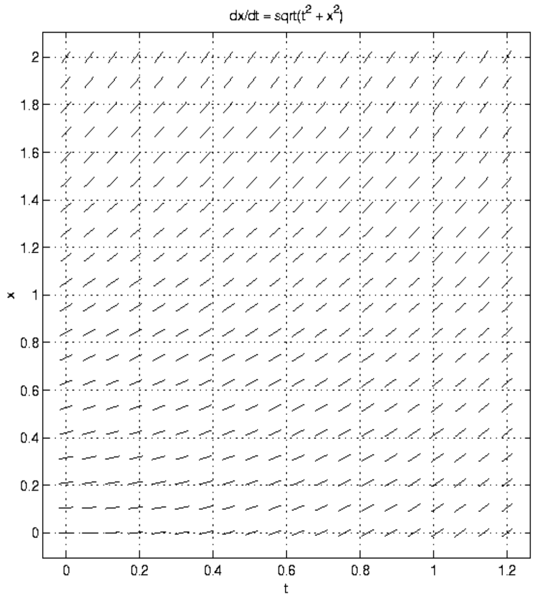
at $t = 0.25, \quad 0.5, \quad 0.75, \quad 1.$

Solution

The results are best presented in the form of a table:

n	t_n	y_n	$f(t_n, y_n)$	$y_n + hf(t_n, y_n)$
0	0.0	0.75	0.75	0.9375
1	0.25	0.9375	0.9703	1.1801
2	0.50	1.1801	1.2816	1.5005
3	0.75	1.5005	1.6775	1.9198
4	1.00	1.9198		

We can compare this approximation with the solution sketched using the direction field:



<u>Important ideas/words from today</u>
Drawing slope fields
Sketching solutions using slope fields
Special cases of slope fields
Euler's method approximates solutions to an IVP and is based on considering slope fields.