Maths 260 Lecture 32

Topics for today
More nonhomogeneous higher order DEs
Forcing and resonance in the harmonic oscillator

Reading for this lecture
BDH Sections 4.3, 4.4

Suggested exercises
BDH Section 4.3; 3, 7, 10, Section 4.4; 2,

Reading for next lecture
BDH Section 6.1

Today's handouts
Lecture 32 notes
Tutorial 9 questions

More on particular solutions to nonhomogeneous DEs

We can modify the method of undetermined coefficients to solve DEs for which the forcing function is a finite sum of terms. For example, to find a solution to

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f_1(t) + f_2(t)$$

we use linearity, i.e., first find y_1 that solves

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f_1(t)$$

and then find y_2 that solves

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f_2(t).$$

Then $y = y_1 + y_2$ solves the DE with forcing term $f(t) = f_1(t) + f_2(t)$.

Example : Find the general solution to

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 2\sin t - e^{2t}$$

Example : Find the general solution to

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = t + 2e^t$$

The forced harmonic oscillator

A forced higher order equation of special interest in applications is the periodically forced harmonic oscillator, i.e.,

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos \omega t$$

for constants $p \geq 0$, $q > 0$ and $\omega > 0$. We are interested in the longterm behaviour of solutions for various values of the damping coefficient p.

The characteristic polynomial is

$$\lambda^2 + p\lambda + q = 0$$

which has roots

$$\lambda = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

Thus, in the case $0 \leq p^2 < 4q$, the homogeneous equation has general solution

$$y_c = c_1 e^{-\frac{p}{2}t} \cos \frac{\sqrt{4q - p^2}}{2} t + c_2 e^{-\frac{p}{2}t} \sin \frac{\sqrt{4q - p^2}}{2} t.$$

<u>Exercise</u> : Find the general solution to the homogeneous problem when $p^2 \geq 4q$.

<u>Exercise</u> : Show that for all values of $p > 0$ the solution to the homogeneous equation tends to zero as $t \to \infty$.

We see that as $t \to \infty$, all solutions to the forced harmonic oscillator with nonzero damping behave the same, i.e., like the particular solution to the non-homogeneous problem. To find the particular solution :

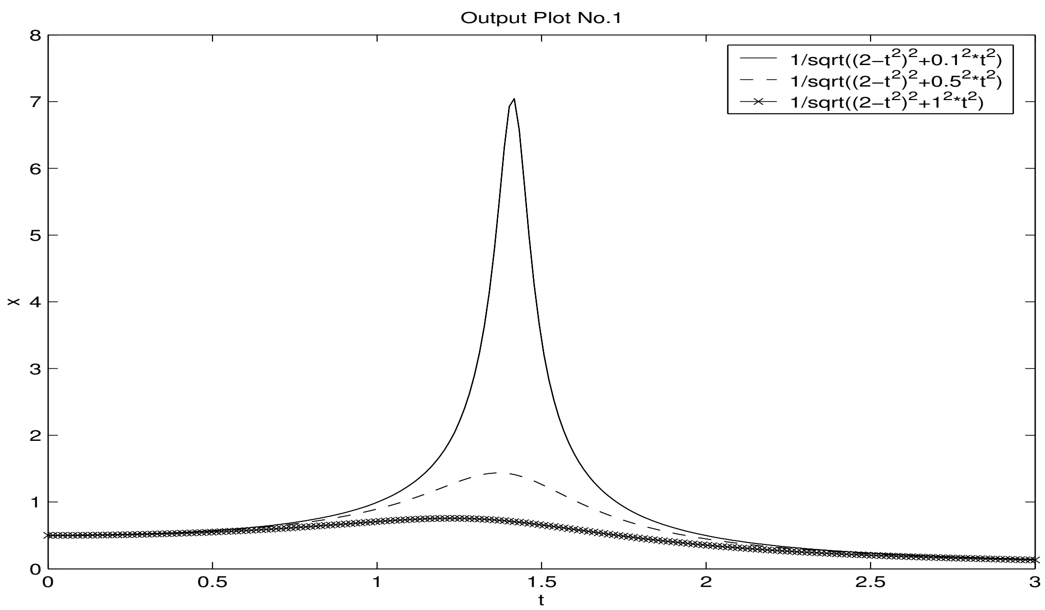
We find $y_p = A\cos(wt + \theta)$ where

$$A = \sqrt{\frac{1}{(q-\omega^2)^2 + (\omega p)^2}},$$

$$\theta = \tan^{-1}\left(\frac{-\omega p}{q-\omega^2}\right)$$

Qualitative behaviour of this solution :

Amplitude of y_p as a function of ω in the case $q = 2$ and various p :



In the undamped case ($p = 0$), a particular solution is

$$y_p = \begin{cases} \frac{1}{q-\omega^2}\cos\omega t, & \omega^2 \neq q \\ \frac{1}{2\omega}\, t\sin\omega t, & \omega^2 = q. \end{cases}$$

<u>Exercise</u> : Check this.

<u>Qualitative behaviour of this solution</u> :

Summary: For the damped periodically forced harmonic oscillator

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos\omega t$$

solutions eventually become periodic with frequency the same as the forcing frequency, ω, and with amplitude depending on ω. The amplitude of the long term solutions can be very large if the forcing frequency (ω), is close to the natural frequency of the unforced system ($\sqrt{q}$). The smaller the damping (p), the closer the frequency of the long term solution is to the natural frequency, and the larger the amplitude of the long term solutions.