Maths 260 Lecture 31

Topic for today
Nonhomogeneous higher order DEs

Reading for this lecture
BDH Section 4.1, 4.2

Suggested exercises
BDH Section 4.1; 1, 3, 7, 11, Section 4.2; 1, 3, 9, 13

Reading for next lecture
BDH Section 4.3

Today's handout
Lecture 31 notes

Nonhomogeneous higher order linear DEs

An nth order linear DE of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

is called nonhomogeneous. The function $f(t)$ is called the forcing function or nonhomogeneous term.

Example :

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = \sin t$$

models the behaviour of a mass/spring system subject to periodic forcing.

To solve a nonhomogeneous DE, we first solve the corresponding homogeneous equation and then combine this solution with a particular solution to the nonhomogeneous equation. This result uses the following:

Extended Linearity Principle:
Given the nonhomogeneous DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

consider the corresponding homogeneous equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

1. If y_h is a solution to the homogeneous DE and y_p is a solution to the nonhomogeneous DE then $y_h + y_p$ is also a solution to the nonhomogeneous DE.

2. If y_c is the general solution to the homogeneous DE and y_p is a solution to the nonhomogeneous equation then $y = y_c + y_p$ is the general solution to the nonhomogeneous DE.

Verification of extended linearity principle for $y'' + py' + qy = f(t)$

Example: Show that $y_p = -2e^{-3t}$ is a solution to the equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2e^{-3t}$$

Hence find the general solution to this equation.

Summary of method of solution for nonhomogeneous equations

1. Find the general solution to the related homogeneous equation.
2. Find **one** solution to the non-homogeneous equation.
3. Add answers to (1) and (2) to get the general solution to the nonhomogeneous equation.
4. If trying to solve an IVP, use the initial conditions to determine constants in the general solution.

Finding a particular solution

We saw (in computer demonstrations) that when the harmonic oscillator is subjected to external forcing, solutions frequently mimic the forcing, at least in the long term. We use this observation as the basis of a method for finding particular solutions to linear, constant coefficient DEs.

Example 1 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^t$$

Example 2 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t$$

Example 3 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t^2$$

Example 4 : Find a solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$$

We can formalise the guessing method used in the examples as:

Method of undetermined coefficients
To find a particular solution to the DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

where $f(t)$ is

(i) a constant, or

(ii) t^n for n a positive integer, or

(iii) $e^{\lambda t}$ for real $\lambda \neq 0$, or

(iv) $\sin bt$ or $\cos bt$, b constant, or

(v) a finite product of terms like (i)-(iv),

take the following steps:

1. Form the UC set consisting of f and all linearly independent functions obtained by repeated differentiation of f.

2. If any of the functions in the UC set is also a solution to the homogeneous DE, multiply all functions in the set by t^k, where k is the smallest integer so that the modified UC set does not contain any solutions to the homogeneous DE.

3. Find a particular solution to the DE by taking a linear combination of all the functions in the (possibly modified) UC set. Determine the unknown constants by substituting this linear combination into the DE.

Example: Find the general solution to

$$\frac{d^2 y}{dt^2} + 4\frac{dy}{dt} + 3y = t$$

Describe the long term behaviour of solutions.

Example: Find the solution to the IVP

$$y'' - 2y' - 3y = te^{3t},$$

$$y(0) = 0, \quad y'(0) = 1$$