# Maths 260 Lecture 30

**Topic for today**
Linear, constant coefficient, higher order DEs
IVPs for higher order DEs
The harmonic oscillator

**Reading for this lecture**
BDH Section 3.6 again

**Suggested exercises**
BDH Section 3.6; 1,3,5,7,9,11

**Reading for next lecture**
BDH Sections 4.1, 4.2

**Today's handout**
Lecture 29 notes

## More on Linear, Constant Coefficient, Higher Order DEs

Consider the differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

Let $y_1(t)$, $y_2(t), \ldots, y_n(t)$ be $n$ linearly independent solutions of the DE. Then

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t)$$

for arbitrary constants $c_i$, is called the **general solution** to the DE. Every solution to the DE can be written in this form by picking the $c_i$ appropriately.

**Example:** Find the general solution to the differential equation

$$2\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 3y = 0$$

**Example:** Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0$$

**Example:** Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$

**General Method:** To find the general solution to

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

1. Write down the **characteristic polynomial**:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots a_1 \lambda + a_0 = 0$$

   and find $n$ roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ (some may be repeated or complex). All functions of the form $e^{\lambda_i t}$, where $\lambda_i$ is a root of the characteristic polynomial, will be solutions to the DE.

2. If all roots are distinct, can construct the general solution by taking a linear combination:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}$$

   (converting to real form if necessary).

3. If a root (say $\lambda_i$) is repeated $k$ times, then the functions

$$e^{\lambda_i t}, t e^{\lambda_i t}, t^2 e^{\lambda_i t}, \ldots, t^{k-1} e^{\lambda_i t}$$

   are linearly independent solutions and we can use a linear combination of these in the general solution.

Remember that the general solution to an $n$th order linear, constant coefficient DE contains $n$ arbitrary constants and $n$ linearly independent solutions.

**Example:** Find the general solution to

$$\frac{d^3 y}{dt^3} + 3 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} = 0$$

**Example:** Find the general solution to

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} = 0$$

**IVPs for Higher Order DEs**

Consider a higher order DE such as

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

with associated system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{Y}$$

where $\mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$ and $v = \dfrac{dy}{dt}$. To define an IVP for the system we specify an initial condition $\mathbf{Y}(t_0) = \mathbf{Y}_0$, i.e., $y(t_0) = y_0$ and $v(t_0) = \dfrac{dy}{dt}(t_0) = v_0$. The equivalent IVP for the original higher order DE therefore has **two** initial conditions: $y(t_0) = y_0$ and $\dfrac{dy}{dt}(t_0) = v_0$.

More generally, an $n$th order IVP is formed from an $n$th order DE

$$a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0y = 0$$

together with $n$ initial conditions $y(t_0) = y_0, \dfrac{dy}{dt}(t_0) = y_1, \ldots \dfrac{d^{n-1}y}{dt^{n-1}}(t_0) = y_{n-1}$.

**Example:** Find a solution to the IVP

$$y'' - 2y' + 10y = 0, \quad y(0) = 0, \ y'(0) = -2$$

Here (and elsewhere) $y' \equiv \dfrac{dy}{dt}, \ y'' = \dfrac{d^2y}{dt^2}$

**The Harmonic Oscillator**

Consider the second order, linear, constant coefficient DE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

where $m, k > 0$, $b \geq 0$. A physical system modelled by this equation is called a **harmonic oscillator**. For instance, the mass/spring system considered in the last lecture is a harmonic oscillator if we assume linear damping and restoring forces, and no external forcing. We can now completely classify the different types of solution to this problem.

Note that the equivalent system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic polynomial is $m\lambda^2 + b\lambda + k = 0$ which has roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, \lambda_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

and the general solution is $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. There are four different cases, depending on the size of $b$, the damping coefficient.

Case 1: $b = 0$ (no damping)

Case 2: $0 < b < \sqrt{4km}$ (underdamped)

Case 3: $b > \sqrt{4km}$ (overdamped)

Case 4: $b = \sqrt{4km}$ (critical damping)

**Summary:**
For the harmonic oscillator, modelled by the DE

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

with constants $b \geq 0$ and $k > 0$ :

- if $b = 0$ all solutions are periodic except the equilibrium at $x = 0$;
- if $b > 0$ all solutions tend to zero as $t \to \infty$.