Maths 260 Lecture 29

Topic for today
Higher order differential equations

Reading for this lecture
BDH Section 3.6

Today's handout
Lecture 28 notes

Chapter 3: Higher Order Differential Equations

Section 3.1 Introduction

Example : Modelling a mass/spring system

We wish to model the motion of an object that is attached to a spring and slides in a straight line on a table.

Let $y(t)$ =position of object at time t with $y = 0$ corresponding to the spring being neither stretched nor compressed.

Main idea from physics : Newton's second law says mass $\times$ acceleration = sum of forces

Typical forces on the object that we might consider are

1. restoring force (spring does not like to be compressed or stretched);

2. frictional forces;

3. external forcing.

Substituting into Newton's law, we get

$$m\frac{d^2y}{dt^2} = r(y) + f(v) + g\,(t, y)$$

where $r(y)$ represents the restoring force at position y, $f(v)$ gives the frictional forces at velocity $v = \frac{dy}{dt}$, $g\,(t, y)$ models any external forcing, and m is the mass of the object attached to the spring.

A common case assumes linear restoring force (i.e., $r(y) = -ky$ for some constant $k > 0$), linear damping (i.e., $f(v) = -bv$ for some constant $b > 0$), and no spatial dependence in the forcing (i.e., g a function of t but not y). The first two assumptions may be valid if y and $v = \frac{dy}{dt}$ remain small. We can write this case as

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{1}{m}g(t)$$

This differential equation is an example of a *higher order differential equation*, i.e., a DE involving derivatives of second or higher order.

Other examples of higher order DEs :

1. $$\frac{d^2\theta}{dt^2} + c_1\frac{d\theta}{dt} + c_2\sin\theta = 0$$

2.

$$\frac{d^3y}{dt^3} - 2y\left(\frac{d^2y}{dt^2}\right)^2 + \frac{dy}{dt} = \sin t$$

3.

$$\begin{aligned}\frac{dx}{dt} &= 2x + y \\ \frac{d^2y}{dt^2} &+ \left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + 3x = 0\end{aligned}$$

We can usually convert a higher order DE into an equivalent system of first order DEs. To do so, define new dependent variables as in the following examples.

Examples:

1.

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

2.

$$\frac{d^3x}{dt^3} + 2\left(\frac{dx}{dt}\right)^2 = \sin t$$

Saying that a system of DEs is *equivalent* to a higher order DE means that if we know a solution to the system we can find one for the higher order equation, and vice versa.

Example : The function

$$y_1(t) = \sin\sqrt{\frac{k}{m}}t$$

is a solution to

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

The pair of functions

$$\left(y_1(t), \frac{dy_1}{dt} = v_1(t)\right) = \left(\sin\sqrt{\frac{k}{m}}t, \sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t\right)$$

is a solution to the equivalent system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}y \end{aligned}$$

To determine the behaviour of solutions of a higher order DE we can rewrite the DE as the equivalent first order system. Then we can study the system using the numerical methods and qualitative techniques (e.g., sketching solutions via phase plane methods) already learnt. We can also use results like the Existence and Uniqueness Theorem. However, in some special cases, it is convenient to study the original higher order equation directly. For example, convenient analytic techniques exist for solving linear higher order equations (see next few lectures).

§3.2 Linear, Constant Coefficient, Higher Order DEs

A differential equation of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

where all a_i are constant, and $a_n \neq 0$, is called an nth order, linear, constant coefficient DE.

Example :

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

Could solve this by converting to a system, then finding eigenvalues and eigenvectors etc. In this section, find a short cut for solving equations of this form.

For previous example, equivalent system is:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} y \\ z \end{pmatrix}$$

Expect solutions of the form $\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$. First component of such a $\mathbf{Y}$ is $y(t) = ce^{\lambda t}$, for some constant c. Hence, guess a solution to the higher order DE of the form $y = e^{\lambda t}$ where λ is to be determined.

Substitute this candidate solution into our DE:

To find a solution to the associated system,

This is exactly what we would have got by using eigenvalues and eigenvectors to solve the system directly. This 'guessing' method is usually shorter than solving the system directly.

Summary:
A higher order differential equation can usually be rewritten as an equivalent system of first order differential equations. Solutions can then be investigated using the methods (qualitative, analytic, numerical) already studied for systems. However, in the case of linear, constant coefficient higher order equations it is usually possible and quicker to find analytic solutions directly. The 'guessing' method we use will be formalised in the next lecture.