Maths 260 Lecture 26

Topics for today
More on linearisation in nonlinear systems
Nullclines

Reading for this lecture
BDH Sections 5.1, 5.2

Suggested exercises
BDH Section 5.2; 1, 5, 7, 9, 11

Reading for next lecture
BDH Section 5.2

Today's handouts
Lecture 25 notes
Tutorial 8 questions

Classification of equilibria in nonlinear systems

Consider a nonlinear system with an equilibrium solution.

1. The equilibrium is a **sink** if all solutions that start close to the equilibrium stay close to the equilibrium for all time and tend to the equilibrium as t increases.

2. The equilibrium is a **source** if all solutions that start close to the equilibrium move away from the equilibrium as t increases.

3. The equilibrium is a **saddle** if there are curves of solutions that tend towards the equilibrium as t increases and curves of solutions that tend towards the equilibrium solution as t decreases. All other solutions started near the equilibrium move away from the equilibrium as t increases and decreases.

To determine the type of an equilibrium in a nonlinear system, can sometimes use linearisation, i.e., use a linear system to approximate the behaviour of solutions near an equilibrium in a nonlinear system. For most systems, knowledge of behaviour of solutions in the linearised system is sufficient to determine the behaviour near the corresponding equilibrium in the nonlinear system. In particular, for the system

$$\frac{d\mathbf{Y}}{dt} = f(\mathbf{Y})$$

with an equilibrium $\mathbf{Y}(t) = \mathbf{Y}_0$, construct the linearised system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}(\mathbf{Y}_0)\mathbf{Y}$$

where $\mathbf{J}(\mathbf{Y}_0)$ is the Jacobian matrix of partial derivatives evaluated at $\mathbf{Y}_0$. If in the linearised system the equilibrium at the origin is a sink, source, or saddle, then $\mathbf{Y} = \mathbf{Y}_0$ is a sink, source, or saddle (respectively) in the nonlinear system.

Sink: the real parts of all eigenvalues are negative.
Source: the real parts of all eigenvalues are positive.
Saddle: some real parts are positive, others negative.
Spiral: some eigenvalues are complex with non-zero real part.
Center: the eigenvalues are purely imaginary.

Note 1: A spiral is always also a saddle, source or sink.

Note 2: linearisation does not tell us anything about the behaviour of solutions to a non-linear system far from an equilibrium.

Unfortunately, linearisation does not always work. In particular, if the Jacobian matrix has a **zero** eigenvalue or a **purely imaginary** eigenvalue, then we cannot predict the behaviour in the nonlinear system based on linearisation alone.

Example:

$$\frac{dx}{dt} = -x^3$$
$$\frac{dy}{dt} = -y + y^2$$

Equilibria:

Jacobian:

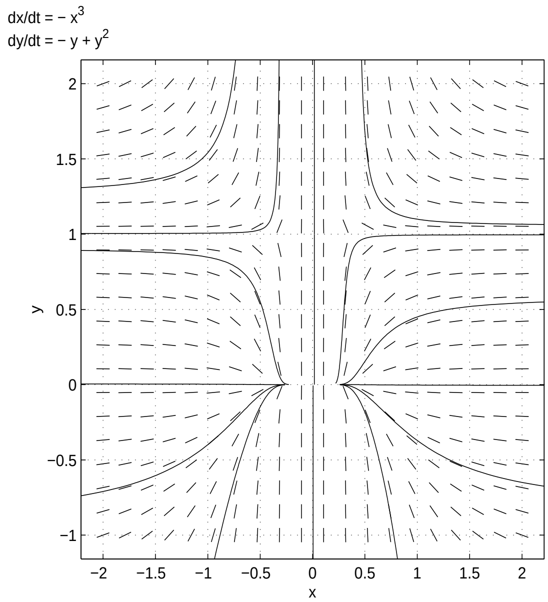
so at $(0, 0)$ the Jacobian is:

Linearised system has phase portrait

At $(0, 1)$ Jacobian is:

Linearised system has phase portrait

However, phase portrait for the nonlinear system is



Notice that in this phase portrait, $(0, 0)$ looks like a sink and $(0, 1)$ looks like a saddle. These results were not predicted by the corresponding linearised systems. Linearisation does not work in these cases because of the zero eigenvalues of the Jacobians.

Sketching phase portraits for nonlinear systems

We would like to be able to sketch the complete phase portrait for a nonlinear system. Linearisation gives us good information about the behaviour of solutions near most equilibria. Can use numerics to fill in the gaps – but it would be helpful to know in advance which regions of the phase space to look at numerically. In order to do this we can use **nullclines**.

Definition: Nullclines
Consider a system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

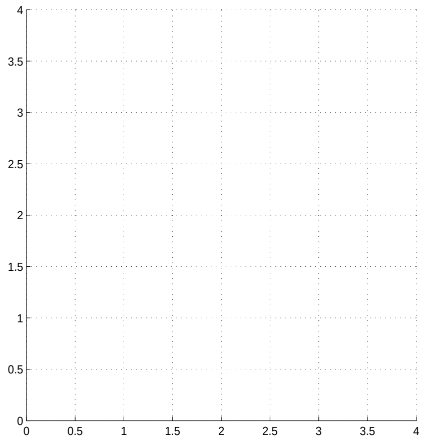
For this system, the x-**nullcline** is the set of points (x, y) where $f(x, y) = 0$. The y-**nullcline** is the set of points (x, y) where $g(x, y) = 0$.

On the x-nullcline, $\frac{dx}{dt} = 0$ and the vector field is vertical, pointing straight up or straight down. On the y-nullcline, $\frac{dy}{dt} = 0$ and the vector field is horizontal, pointing either left or right. At the intersections of the x- and y-nullclines, $f(x, y) = g(x, y) = 0$, i.e., a point of intersection between an x-nullcline and a y-nullcline is an equilibrium.

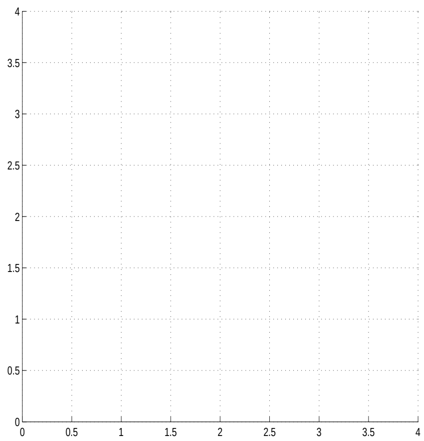
Example: Use nullclines to sketch the phase portrait for the system

$$\begin{aligned}\frac{dx}{dt} &= x(2 - x - y),\\ \frac{dy}{dt} &= y(3 - 2y - x), \quad x, y \geq 0\end{aligned}$$

The x-nullclines are:

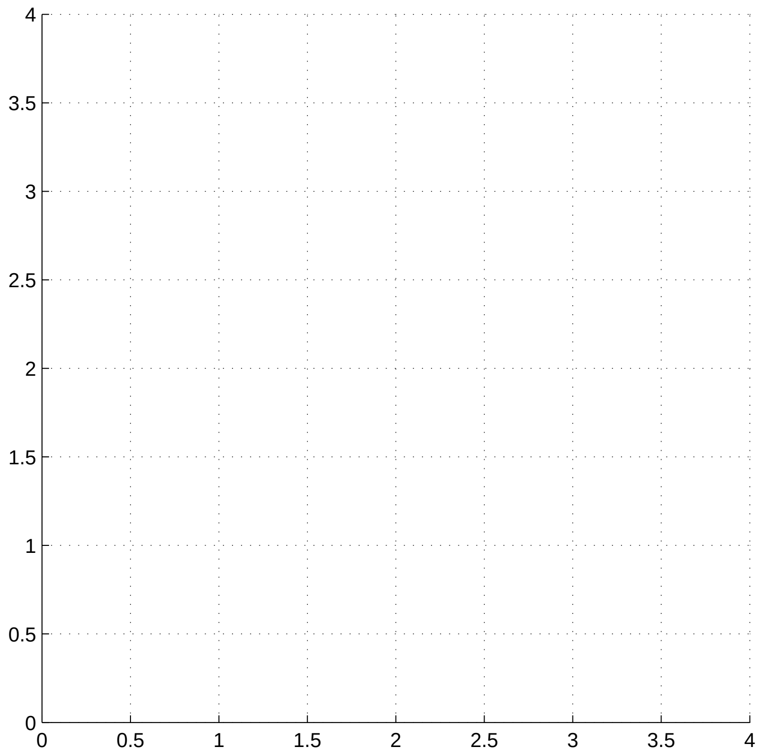


The y-nullclines are:



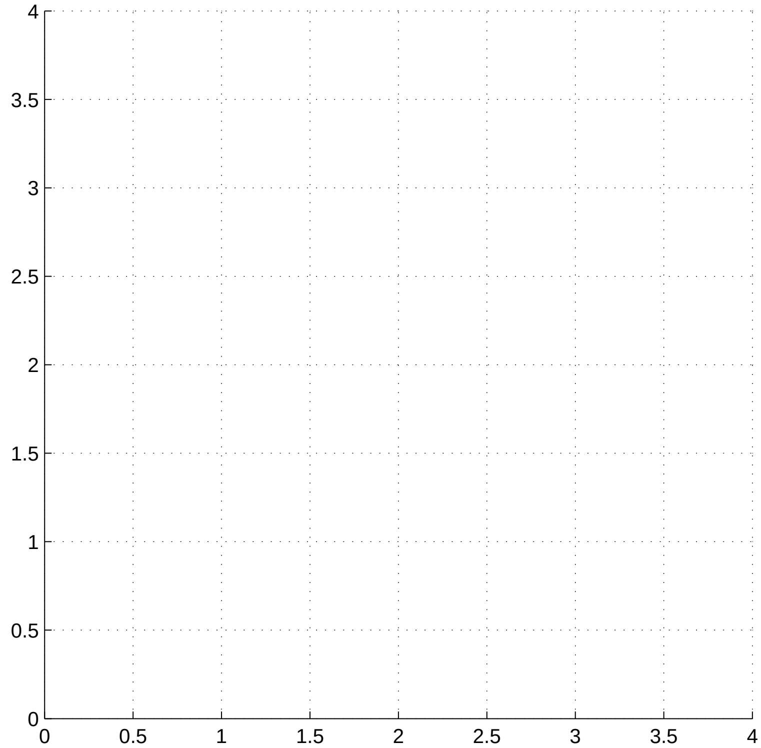
Nullclines divide the phase plane into regions where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ have constant sign. In example above:

Combining information about x- and y-nullclines for this example, we get



Now the phase plane is divided into four regions:

We use the known direction of solution curves in each region to determine the direction of solutions on the nullclines.



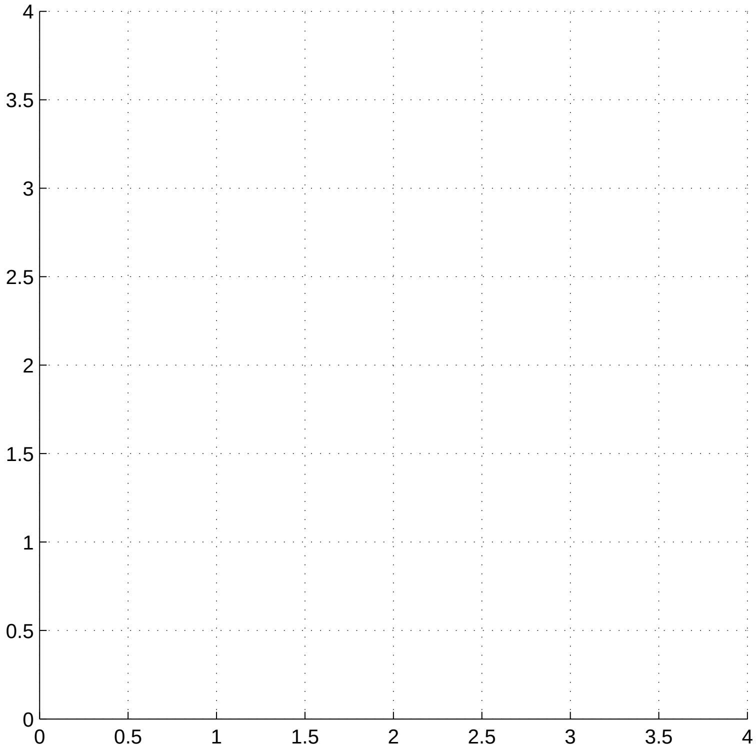
Now we can see that:

1. Once solutions get into region B they cannot get out again. Solutions move down and right until they get to lower right corner (i.e., equilibrium at $(1, 1)$).

2. Similarly, once solutions get into region D they cannot get out again. Solutions move up and left until they get to upper left corner (i.e., equilibrium at $(1, 1)$).

3. Solutions starting in region A or C must either leave the region by entering B or D (and then tend to $(1, 1)$) or must tend directly to $(1, 1)$.

Hence, the phase portrait for the system must be, approximately:



This picture suggests that $(1, 1)$ is a sink, $(0, 0)$ is a source, $(2, 0)$ and $(0, 3/2)$ are saddles. Linearisation confirms that this is so.

The approximate phase portrait obtained using nullclines looks very like the phase portrait obtained with *pplane*.