Maths 260 Lecture 25

Topic for today
Non-linear systems: linearisation near equilibria

Reading for this lecture
BDH Section 5.1

Suggested exercises
BDH Section 5.1; 1, 3, 7, 9, 11

Reading for next lecture
BDH Section 5.2

Today's handout
Lecture 24 notes

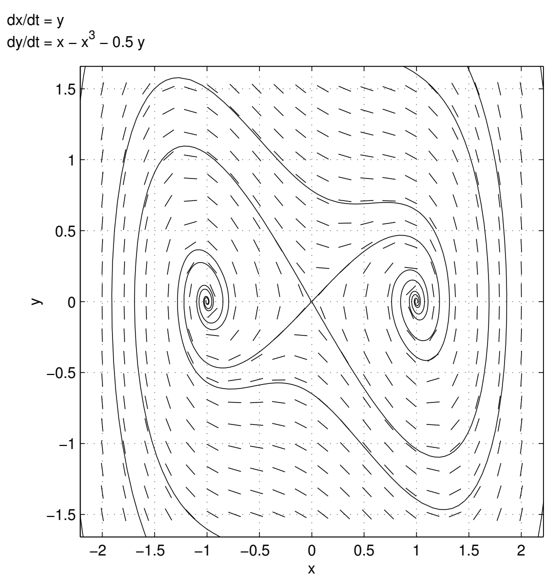
2.10 Nonlinear Systems

Consider the system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^3 - \frac{1}{2}y\end{aligned}$$

Equilibrium solutions:

Slope field and some solutions



We can understand the saddle-like nature of $(0,0)$ if we approximate the nonlinear system by a linear system.

For x, y very close to zero, x^3 is much smaller than x or y.

So we can ignore x^3 term in the nonlinear system, and approximate the behaviour of the nonlinear system near $(0,0)$ with the linear system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - \frac{1}{2}y \end{aligned}$$

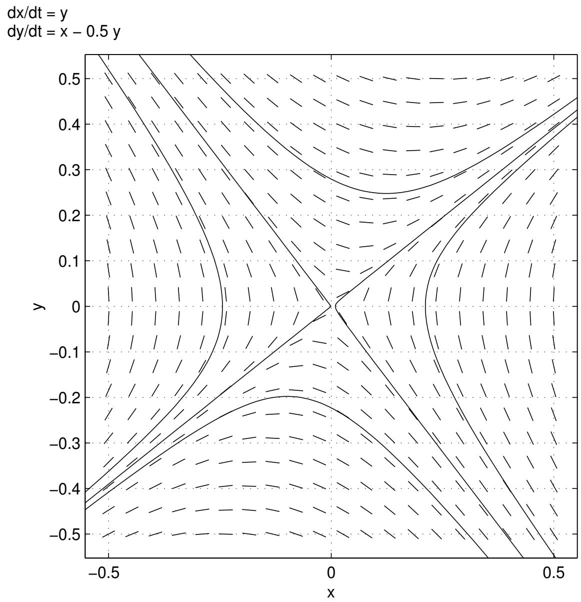
i.e.,

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

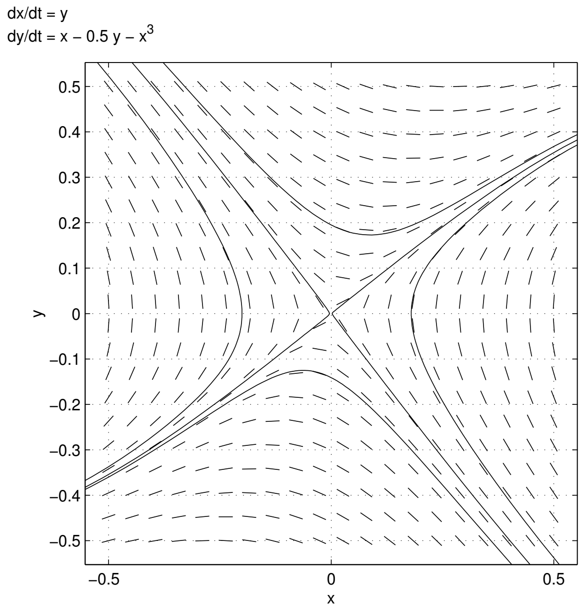
The eigenvalues of matrix $\mathbf{A}$ are 0.78 and -1.28 so the equilibrium at the origin of linear system is a saddle.

The following pictures show the slope field and solutions for the linear system and for the nonlinear system near the origin. Note that the linear system is a good approximation near the origin but is hopeless away from the origin.

Slope field and solutions for linear system



Slope field and solutions for nonlinear system



This procedure is called **linearisation**: Near an equilibrium, approximate the nonlinear system by an appropriate linear system. For initial conditions near the equilibrium, solutions of the nonlinear system stay close to solutions of the approximate linear system, at least for some interval of time. Thus, the type of equilibrium at the origin in linearised system gives information about the type of the corresponding equilibrium in the nonlinear system.

Returning to original example, consider equilibria at $(1,0)$ and $(-1,0)$.

To approximate behaviour near $(1,0)$ by a linear system, we need to first shift the equilibrium to the origin – because linear systems usually only have an equilibrium at the origin.

Change the coordinates as follows:

$$\begin{aligned} u &= x - 1 \\ v &= y \end{aligned}$$

so the equilibrium $(x,y) = (1,0)$ is now at $(u,v) = (0,0)$. Then the system becomes:

$$\frac{du}{dt} =$$

$$\frac{dv}{dt} =$$

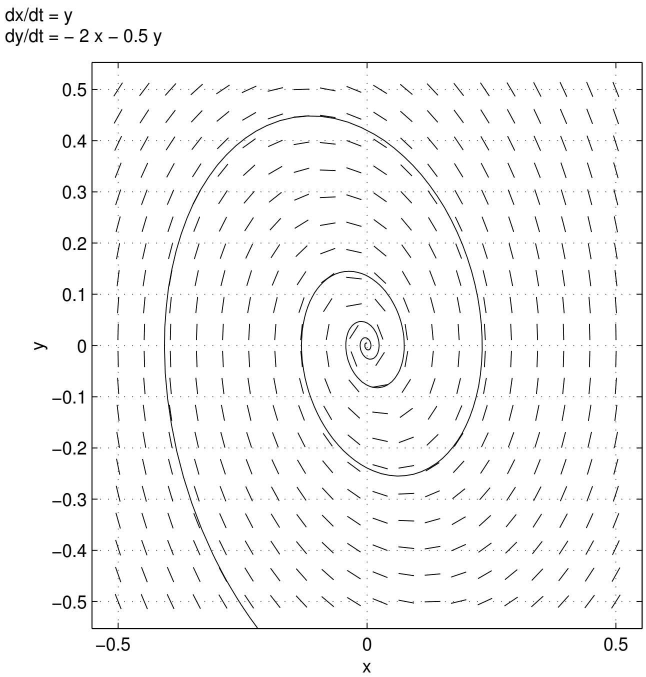
For u and v small, $-3u^2$ and $-u^3$ are very, very small. Ignore these nonlinear terms and approximate system by:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

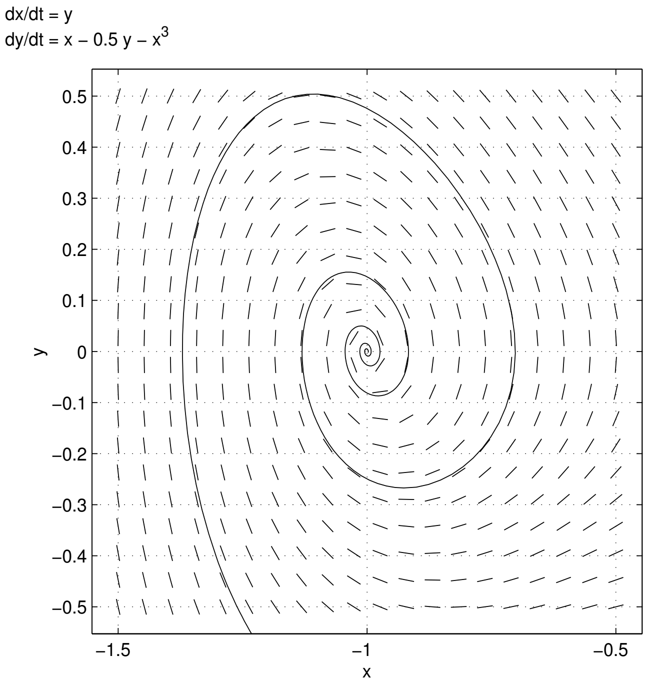
Eigenvalues are $-\frac{1}{4} \pm \frac{1}{4}\sqrt{31}i$. So origin is a spiral sink in the linear approximation.

The following pictures illustrate the similarity between the phase portrait near the equilibrium at $(1, 0)$ in the nonlinear system and the phase portrait for the linearised system.

Phase portrait for linearised system



Phase portrait near $(1, 0)$ in nonlinear system



Similar calculations give similar results for the equilibrium at $(-1, 0)$.

More generally, if the system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

has an equilibrium at (x_0, y_0), we can construct a linear approximation to the system for x and y values near (x_0, y_0) as follows:

First move the equilibrium to the origin. Write $u = x - x_0, v = y - y_0$. The nonlinear equations in the new coordinates are:

$$\begin{aligned}\frac{du}{dt} &= \frac{dx}{dt} = f(x,y) = f(x_0+u, y_0+v)\\ \frac{dv}{dt} &= \frac{dy}{dt} = g(x,y) = g(x_0+u, y_0+v)\end{aligned} \tag{1}$$

Now we use Taylor expansion to rewrite f and g:

$$f(x_0+u, y_0+v) = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right] u + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right] v + h.o.t$$

$$g(x_0+u, y_0+v) = g(x_0,y_0) + \left[\frac{\partial g}{\partial x}(x_0,y_0)\right] u + \left[\frac{\partial g}{\partial y}(x_0,y_0)\right] v + h.o.t$$

If we ignore the higher order terms and note that $f(x_0, y_0) = g(x_0, y_0) = 0$, then we get an approximate linear system:

$$\begin{pmatrix} \frac{du}{dt} \\ \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} u \\ \\ v \end{pmatrix} \tag{2}$$

i.e., the behaviour of solutions to the nonlinear system near the equilibrium (x_0, y_0) can be approximated by the behaviour of solutions in the linearised system (2).

The matrix of partial derivatives in (2) is called the **Jacobian** matrix.

Example: Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x(1+x^2) \\ \frac{dy}{dt} &= 3y(1-y-x) \end{aligned}$$

Find the equilibria and determine their types.

The phase portrait for this system, drawn with *pplane*, is given below. Note the source at $(0,0)$ and the saddle at $(0,1)$ as predicted by our calculations.