Maths 260 Lecture 24

Topics for today
Bifurcations in linear systems

Reading for this lecture
BDH Section 3.7

Suggested exercises
BDH Section 3.7; 2(b,c), 6(b,c)

Reading for next lecture
BDH Section 5.1

Today's handouts
Lecture 23 notes

2.9 Putting it all together - bifurcations in linear systems

Bifurcation: sudden *qualitative* change in the dynamics.

In our examples today, the following results will be useful:

For any matrix A, $\det(A)$ = product of eigenvalues and $\text{trace}(A)$ = sum of eigenvalues. If A is a 2×2 matrix, the signs of $\det(A)$ and $\text{trace}(A)$ tell us a lot about the type of the equilibrium at the origin for the system $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$. For example, if A is a 2×2 matrix with $\det(A) < 0$, then the origin is a saddle.

Example : Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix} \mathbf{Y}$$

where a is a parameter. Determine the type of equilibrium at the origin for all values of a. Sketch the phase portrait for representative values of a.

Eigenvalues of matrix

$$A = \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix}$$

are

$$\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 8a}$$

so the type of equilibrium at the origin depends on a. Also, $\det(A) = -2a$ and $\text{trace}(A) = 1$.

We find the following qualitatively distinct cases, depending on a.

1. If $1 + 8a < 0$, eigenvalues of A are complex, say $\alpha \pm i\beta$.

2. If $1 + 8a > 0$, eigenvalues of A are real. There are two subcases:

   (a) If $a > 0$, $\det(A) < 0$ so there is one positive eigenvalue and one negative eigenvalue.

(b) If $-\frac{1}{8} < a < 0$, $\det(A) > 0$, so eigenvalues are of the same sign (and real). But $\operatorname{trace}(A) > 0$ so both eigenvalues are positive.

3. **Transitional values of a**

(a) $a = -\frac{1}{8}$,

$$A = \begin{pmatrix} 1 & 2 \\ -\frac{1}{8} & 0 \end{pmatrix}$$

In this case, eigenvalues of A are $\frac{1}{2}$ (twice) with just one linearly independent eigenvector

$$\begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

(b) $a = 0$,

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

In this case, eigenvalues of A are 0 and 1 with eigenvectors

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

respectively.

Representative phase portraits from *pplane* are given below.

Example : Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \mathbf{Y}$$

where a is a parameter. Determine the type of equilibrium at the origin for all values of a. Sketch the phase portrait for representative values of a.

From these two examples we see how the transitional cases arise as a parameter is varied:

1. a centre occurs as a spiral sink changes to a spiral source, or vice versa;
2. an improper node (i.e., two equal eigenvalues with only one linearly independent eigenvector) occurs when a spiral sink (or source) turns into a real sink (or source), or vice versa;
3. a linear system with a zero eigenvalue occurs when a saddle turns into a sink or source, or vice versa.