Maths 260 Lecture 23

Topics for today
Linear systems with repeated eigenvalues
Linear systems with zero eigenvalues

Reading for this lecture
BDH Section 3.5

Suggested exercises
BDH Section 3.5; 1, 3, 5, 7, 11, 21

Reading for next lecture
BDH Section 3.7

Today's handouts
Lecture 21 notes
Tutorial 7 questions

2.8 Special Cases of Linear Systems

Linear systems with repeated eigenvalues

Example Consider the system

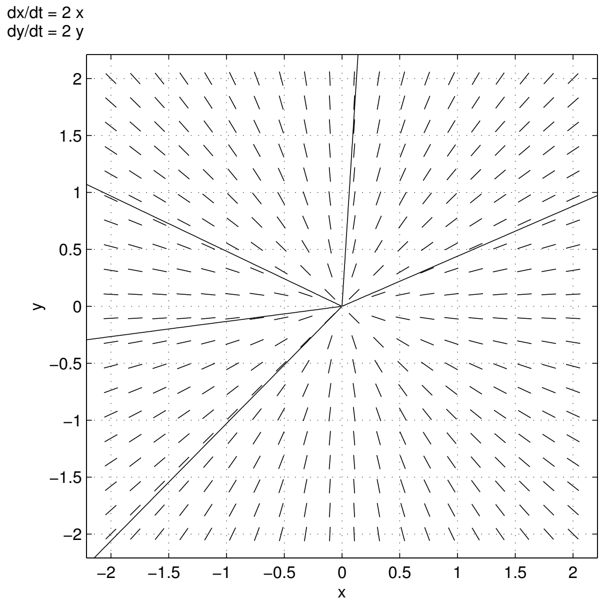
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{Y}$$

Eigenvalues are 2 and 2. Eigenvectors are:

The general solution is:

i.e., every non-zero solution is a straight-line solution.

Phase portrait



This example illustrates a general case: If matrix A has a repeated eigenvalue λ with two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\mathbf{Y}_1 = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{Y}_2 = e^{\lambda t}\mathbf{v}_2$ are linearly independent straight line solutions. We can construct a general solution from a linear combination of these two solutions as usual.

Furthermore, if A is a 2×2 matrix, then every solution except the equilibrium at the origin is a straight-line solution. If $\lambda > 0$, then every non-zero solution tends to ∞ as $t \to \infty$ (so the origin is a source). If $\lambda < 0$, then every solution tends to the origin as $t \to \infty$ (so the origin is a sink).

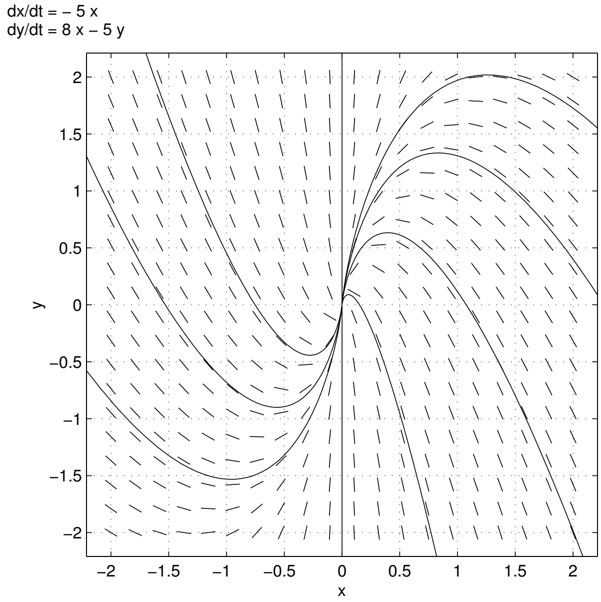
What happens if we cannot find two linearly independent eigenvectors?

Example Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 8 & -5 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues are -5 and -5.
Eigenvectors are:

Phase portrait and some solutions



See that system has only one straight line solution. We can't write the general solution as a linear combination of solutions of the form $e^{\lambda t}\mathbf{v}$ because we don't have enough such solutions. To find a second solution, we use the following result.

Theorem: Consider the system

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$$

where A has a repeated eigenvalue λ with just one linearly independent eigenvector. Pick an eigenvector $\mathbf{v}_1$ corresponding to λ. Then

$$\mathbf{Y}_1 = e^{\lambda t}\mathbf{v}_1$$

is a straight-line solution and

$$\mathbf{Y}_2 = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$$

is a second, linearly independent solution of the system, where $\mathbf{v}_2$ is a vector satisfying

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$$

($\mathbf{v}_2$ is called a generalised eigenvector). Can use this second solution $\mathbf{Y}_2$ to construct the general solution for the previous example.

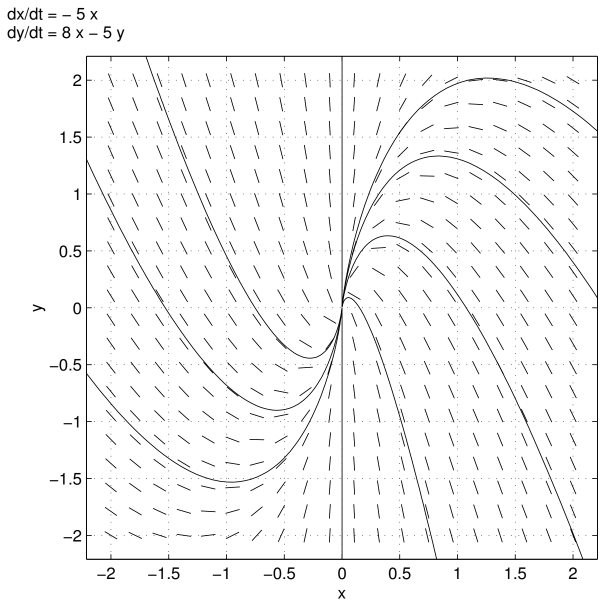
Example

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 8 & -5 \end{pmatrix}\mathbf{Y}.$$

Found already that $Y_1 = e^{-5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a solution. Look for $\mathbf{v}_2$ satisfying

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$$

Phase portrait and some solutions



We see that all solutions are tangent at the origin to the direction of the straight-line solution. This is always the case in a 2×2 system: when there is a non-zero repeated eigenvalue with only one corresponding linearly independent eigenvector, all solution curves in the phase plane are tangent to the straight-line solution.

Important note: There is some freedom when choosing a generalised eigenvector (e.g., in last example,

$$\mathbf{v}_2 = \begin{pmatrix} \frac{1}{8} \\ y \end{pmatrix}$$

is a generalised eigenvector for any choice of y). However, a multiple of a generalised eigenvector **is not** usually a generalised eigenvector (e.g., in last example,

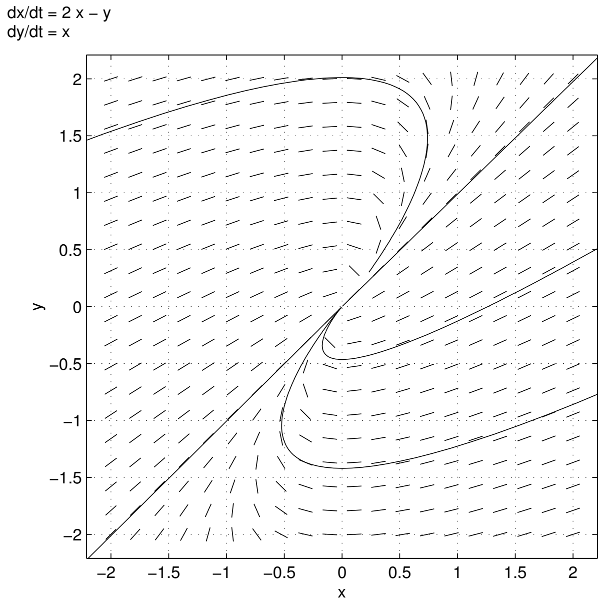
$$k \begin{pmatrix} \frac{1}{8} \\ y \end{pmatrix}$$

is not a generalised eigenvector for any choice of k except $k = 1$). Different choices of the generalised eigenvecetor all lead to the same general solution.

Example : Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

Direction field and some solutions



Linear systems with zero eigenvalues

Example Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \mathbf{Y}$$

Eigenvalues are 5 and 0 with eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. So

$$\mathbf{Y}_1 = e^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and

$$\mathbf{Y}_2 = e^{0t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

are linearly independent solutions, and the general solution is:

If $c_1 = 0$, then

$$\mathbf{Y}(t) = c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

which is constant, so this is an equilibrium solution for all choices of c_2. This is a general result: all points on a line of eigenvectors corresponding to a zero eigenvalue are equilibrium solutions.

If $c_1 \neq 0$ then first term in general solution tends to zero as $t \to -\infty$, i.e., solution tends to the equilibrium

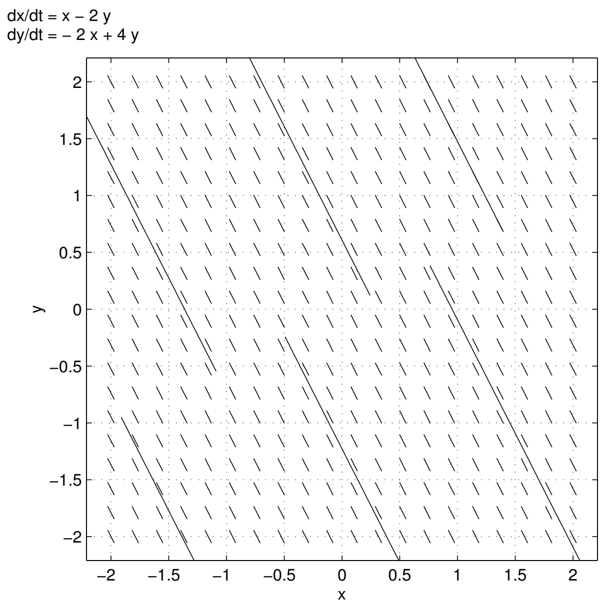
$$c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

along a line parallel to

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

as $t \to -\infty$. Hence, phase portrait is qualitatively:

From *pplane*, get:



Get similar behaviour in other linear systems with a zero eigenvalue, but details of the general solution and the phase portrait may vary depending on the specific example.

Example : Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix} \mathbf{Y}$$