Maths 260 Lecture 22

Topic for today
Linear systems with complex eigenvalues

Reading for this lecture
BDH Section 3.4

Suggested exercises
BDH Section 3.4; 1, 3, 5, 7, 9, 11, 23

Reading for next lecture
BDH Section 3.5

Today's handouts
Lecture 20 notes

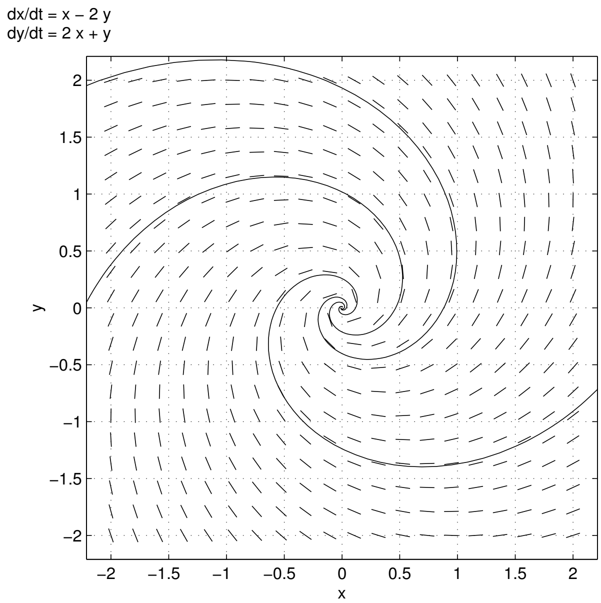
2.8.3 Linear systems with complex eigenvalues

There exist linear systems for which there are no straight-line solutions.

Example: Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

Slope field and some solutions



What goes wrong?

Calculate the eigenvalues:

See that eigenvalues are complex. We saw earlier that straight-line solutions result from real eigenvalues. That is, $\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$ is a solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{v}$ but the corresponding solution curve will not be a straight-line if λ is not real.

Find (complex) solution vectors for this example:

How do we interpret a complex-valued solution? We would like a real-valued solution.

Theorem
Consider the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

If $\mathbf{Y}(t)$ is a complex-valued solution to the system, write

$$\mathbf{Y}(t) = \mathbf{Y_R}(t) + i\mathbf{Y_I}(t)$$

Then $\mathbf{Y_R}(t)$ and $\mathbf{Y_I}(t)$ are real-valued solutions to the system and are linearly independent.

Proof

Apply theorem to previous example. Know that

$$e^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

is a solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\mathbf{Y}.$$

But

$$e^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} =$$

Hence, by theorem,

$$\mathbf{Y_R} =$$

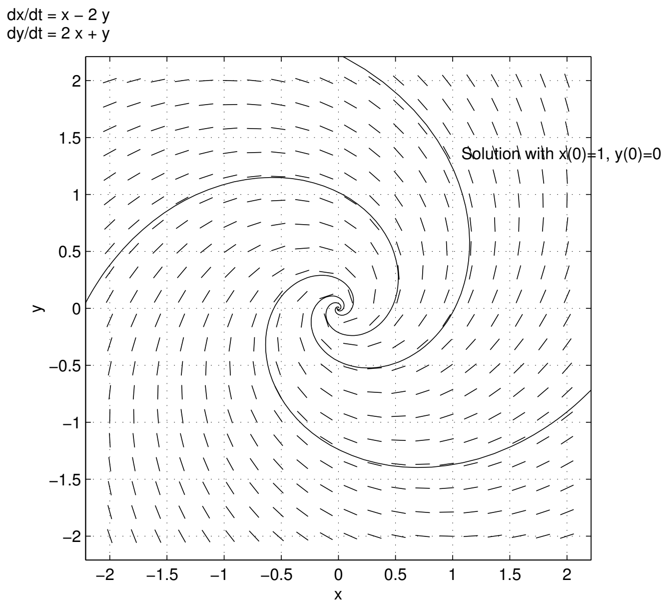
and

$$\mathbf{Y_I} =$$

are real valued, linearly independent solutions and the general solution is

We see from the general solution that each component of $\mathbf{Y}$ will oscillate from positive to negative and that amplitude of each component will grow exponentially.

Phase portrait



Components of solution with $x(0) = 1, y(0) = 0$

Note : In this example, we found two linearly independent real-valued solutions by taking the real and imaginary parts of the complex-valued solution

$$e^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

What if we instead used the real and imaginary parts of the other complex-valued solution we found, i.e.,

$$e^{(1-2i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

We see that the other complex-valued solution also gives us two real-valued solutions but these solutions are just multiples of the real-valued solutions already found. Thus, using the other complex-valued solution gives no new information; we can form the general solution using the real and imaginary parts of just one of the complex conjugate pair of solutions.

In general, the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

with complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ has a solution of the form

$$\mathbf{Y}(t) = e^{(\alpha+i\beta)t}\mathbf{Y}_0,$$

where $\mathbf{Y}_0$ is the eigenvector corresponding to eigenvalue $\lambda_1 = \alpha + i\beta$.
Expanding:

$$\mathbf{Y}(t) = e^{(\alpha+i\beta)t}\mathbf{Y}_0 = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\mathbf{Y}_0.$$

So the general solution is a combination of exponential and trigonometrical terms. The qualitative behavior of solutions depends on α and β.

When $\mathbf{A}$ is a 2×2 matrix, trig terms alternate between positive and negative with period $\frac{2\pi}{\beta}$, so the solution curves spiral around the origin in the phase plane and:

1. If $\alpha > 0$, then $e^{\alpha t} \to \infty$ as $t \to \infty$ so solution curves spiral away from the origin. In this case, the equilibrium at the origin is called a **spiral source**.
   Typical phase portraits:

2. If $\alpha < 0$, then $e^{\alpha t} \to 0$ as $t \to \infty$ so solution curves spiral into the origin. In this case, the equilibrium at the origin is called a **spiral sink**.
   Typical phase portraits:

3. If $\alpha = 0$, then $e^{\alpha t} = 1$ and solution curves are periodic; solutions return to their initial conditions in the phase plane and repeat the same curve over and over again. In this case, the equilibrium at the origin is called a **centre**.
   Typical phase portraits:

Examples

1. Sketch the phase portrait for the system

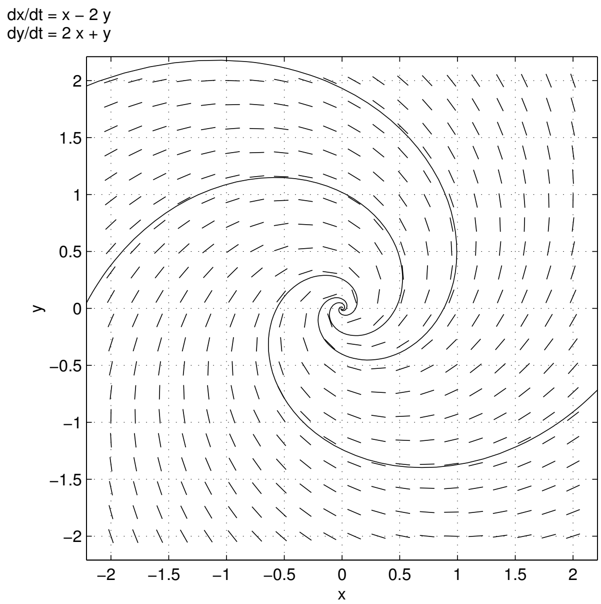
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

As before, e-values are $1 \pm 2i$ ($\alpha = 1$, $\beta = 2$) so origin is a spiral source. To determine whether spiral is clockwise or anticlockwise, evaluate vector field at a point. For example, at $(x, y) = (0, 1)$ on the y-axis, vector field is

$$\mathbf{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

which points to left, so spiral is anticlockwise.

Direction field and some solutions



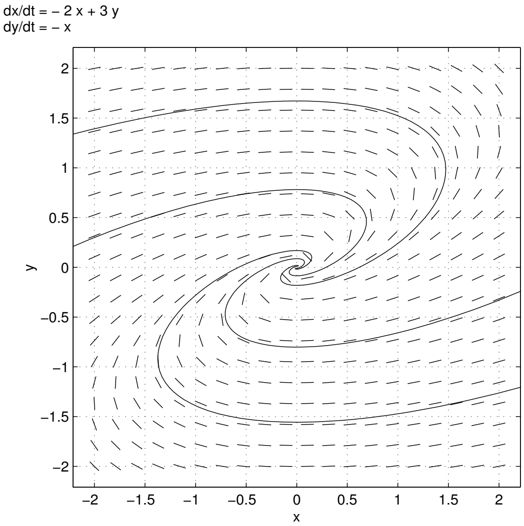
Exercise: Show that the general solution to the system, written in terms of real functions, is

$$Y(t) = c_1 e^t \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

2. Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

Direction field and some solutions



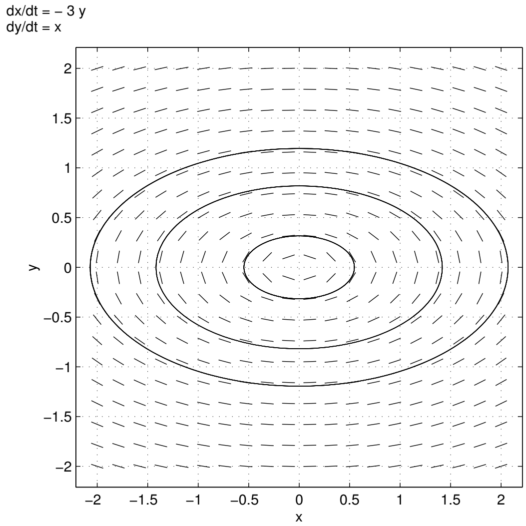
Exercise: Show that the general solution to the system, written in terms of real functions, is

$$\mathbf{Y}(t) = c_1 e^{-t} \begin{pmatrix} \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \\ \cos(\sqrt{2}t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin\sqrt{2}t - \sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t \end{pmatrix}$$

3. Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

Direction field and some solutions



Exercise: Show that the general solution to the system, written in terms of real functions, is

$$\mathbf{Y}(t) = c_1 \begin{pmatrix} 3\cos\sqrt{3}t \\ \sqrt{3}\sin\sqrt{3}t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin\sqrt{3}t \\ -\sqrt{3}\cos\sqrt{3}t \end{pmatrix}$$

4. Find the general solution (expressed in terms of real functions) for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{pmatrix} \mathbf{Y}.$$

Determine the long term behaviour of solutions.

E-values are $1,\ 2+3i,\ 2-3i$ with corresponding e-vectors

$$\begin{pmatrix} -10 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

respectively.

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