Maths 260 Lecture 21

Topics for today

Complex Numbers:

- Multiplication of polar forms
- De Moivre's formula
- Derivatives of complex-valued functions
- Euler's formula
- The exponential of a complex number

Reading for this lecture

Background on complex numbers

BDH Appendix B

Suggested exercises

Problems at the back of the handout on complex numbers.

Reading for next lecture

BDH Section 3.4

Recall: Multiplication of polar forms

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be any two complex numbers, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Hence,

multiplying ⇔ *absolute value* = *product of absolute values*
argument = *sum of arguments*

Example: Calculate

- $(\cos\theta + i\sin\theta)^2 =$
- $(\cos\theta + i\sin\theta)^3 =$

These are particular cases of **de Moivre's formula:**

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta),$$

a very useful formula...

Example: Express $\cos 2\theta, \sin 2\theta$ in terms of $\cos\theta, \sin\theta$.

From the de Moivre's formula, we have

$\cos(2\theta) + i\sin(2\theta) =$

So

Polar forms are sometimes useful for solving equations.

Example Solve $z^3 = 1$.

$z = 1$ is obviously a solution. Any others? Let's write

$$z = r(\cos\theta + i\sin\theta),$$

where $r = |z| > 0$. Then

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

and therefore

$$r^3(\cos 3\theta + i\sin 3\theta) = 1$$

and

Notice that for other values of n, the solutions given coincide with the above solutions because of the periodicity of cos and sin.

Plot the solutions:

Derivatives of complex valued functions

Suppose t is real and $f(t)$ is a complex valued function of t, i.e.

$$f(t) = u(t) + iv(t)$$

Then, if u and v are differentiable at t, we define the derivative of $f(t)$ to be

$$\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}$$

Example Find the derivative of $f(t) = \cos(t) + i\sin(t)$

Properties of $f(t)$:

- $f'(t) = if(t)$,
- $f(0) = 1$,
- $f(t_1)f(t_2) = f(t_1 + t_2)$.

Compare this to $g(t) = e^{at}$, where a is real:

Properties of $g(t)$:

- $g'(t) =$
- $g(0) =$
- $g(t_1)g(t_2) =$

Euler's Formula

The properties of f prompted Euler to make the definition:

Euler's Formula: $e^{it} = \cos t + i\sin t$

Euler's Formula and Polar forms

Example: $z = 1 + i$.

In general, complex number $z = a + ib$ can be written in polar form as

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

Now multiplication and division are easy:

Example: $z_1 = 2e^{i\pi/6}$, $z_2 = -e^{i\pi/4}$.

Also we can easily calculate **powers**:

Example 1: If $z = 3e^{i\pi/5}$, find z^2 and z^5.

Example 2: Find all solutions of $z^3 = 2$.

The Exponential of a Complex Number

We know how to calculate e^x when x is real and e^{iy} when y is real, so it makes sense to define:

Definition: $e^{x+iy} = e^x e^{iy}$

Example: Calculate $e^{\log(2)+i\pi}$.

Example: Show that if λ is a complex number then

$$\frac{d}{dt}\left(e^{\lambda t}\right) = \lambda e^{\lambda t}$$

Example: Find all solutions of the form $y = e^{\lambda t}$ to the differential equation

$$y''(t) + 2y'(t) + 10y(t) = 0.$$

Note: We'll see later that the *general solution* of such equations can be found by taking a linear combination of the real and imaginary parts of the complex exponential solutions. So the general solution of the equation above is

$$y = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t.$$

Example
Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{pmatrix}.$$