Maths 260 Lecture 20

Topic for today
Complex Numbers

Reading for this lecture

Background on complex numbers.
Reading for next lecture

The handout on complex numbers
Today's handouts
Lecture 20 notes

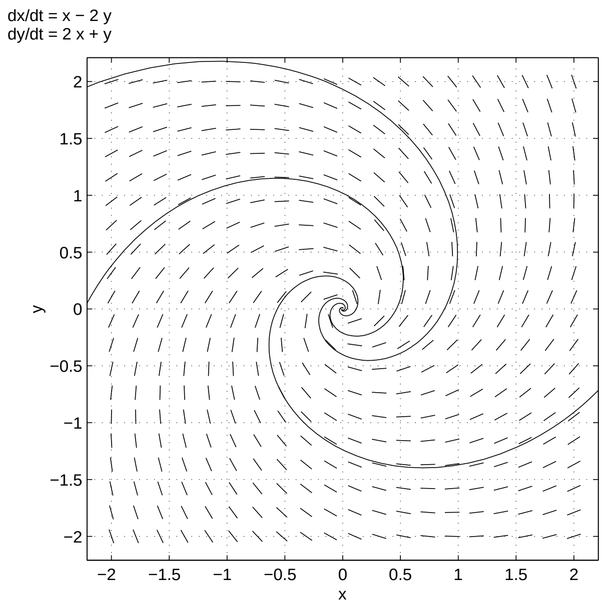
2.8.1 Complex Numbers

In order to apply analytic techniques to systems of DEs, we need to know about complex numbers and their properties.

Example Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

Slope field and some solutions



There are no straight-line solutions! Let's see if we can find out why.
Calculate the eigenvalues:

$$0 = \det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 5.$$

So the quadratic formula gives:

Notes:

- We need the square root of a negative number!
- The matrix doesn't have any eigenvalues that are *real numbers*. That's why there are no straight-line solutions to the system of DEs.
- However, we can still find eigenvalues that are *complex numbers*.
- We *can* still calculate two eigenvalues provided we introduce a new number:

$$i = \sqrt{-1}$$

$\lambda =$

Example Solve $x^2 + 2x + 5 = 0$.

Then solutions to above equation are:

We can use i to get solutions to any quadratic equation

$$ax^2 + bx + c = 0$$

where $b^2 - 4ac < 0$.

In fact, one can prove: **Fundamental Theorem of Algebra:**

An nth degree equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$ has n solutions.
This means that the polynomial can be factorised:

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = a_n(x - x_1)(x - x_2)(x - x_3)\dots(x - x_n),$$

where x_1, x_2, $\dots$, x_n are the roots of the equation. Note that some of these x_i may be repeated.

Definition A complex number is a number of the form

$$z = a + ib$$

where a and b are real and $i^2 = -1$.

Can think of a complex number as a *pair* of real numbers.

Geometric Interpretation - the **Argand Diagram**.

Definitions

- The **complex conjugate** of a complex number $z = a + ib$ is

$$\bar{z} = a - ib$$

- The **real part** of z is Re $z = a$.
- The **imaginary part** of z is Im $z = b$.

Example: $z = 2 - 3i$

$\bar{z} =$
Re $z =$
Im $z =$

Algebra of Complex Numbers

Addition/Subtraction: collect real and imaginary terms.

Example: $(2 + 4i) + (3 - 2i) =$

Example:$(-1 - 4i) - (4 + 3i) =$

Multiplication: Multiply out brackets and collect real and imaginary terms, remembering that $i^2 = -1$.

Example: $(2 + 4i)(3 - 2i) =$

$(-1 - 4i)(4 + 3i) =$

Division: Multiply top and bottom by complex conjugate of denominator, then collect real and imaginary terms.

Example: $\dfrac{2 + 4i}{3 + 2i} =$

Example: $\dfrac{-1 - 4i}{4 - 3i} =$

Polar Form

Another way of describing a given complex number is to use *polar co-ordinates*: the distance of z from 0 and the angle it makes with the real axis.

Note: $a = r\cos\theta$, $b = r\sin\theta$.

So $z = a + ib = r(\cos\theta + i\sin\theta)$.

And so $r = \sqrt{a^2 + b^2}$ (the *modulus* of z), denoted $|z|$,
and $\theta = \tan^{-1}(b/a)$ (the *argument* of z), denoted $\arg z$.

Example Convert $z = 1 + i$ into polar form.

Example Convert $z = 3(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$ into rectangular form.

Multiplication of polar forms

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be any two complex numbers, then

$$\begin{aligned} z_1 z_2 &= \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

Hence, **multiplying** corresponds to

$$\begin{aligned} &\textit{absolute value} = \textit{product of absolute values} \\ &\textit{argument} \ = \textit{sum of arguments} \end{aligned}$$

Picture:

Example
Convert $1+i$ and $2i$ into polar form.

Multiply out $(1+i)2i$, and convert the product to polar form.

Compare with the polar forms of $1+i$ and $2i$.