Maths 260 Lecture 19

Topic for today
Classification of equilibria in linear systems with real eigenvalues

Reading for this lecture
BDH Section 3.3

Suggested exercises
BDH Section 3.2, 1, 5, 9, 11, 19

Reading for next lecture
BDH Section 3.4

Today's handouts
Lecture 19 notes

Section 2.7 Classification of equilibria in linear systems with real eigenvalues

This lecture looks at systems of the form

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY},$$

where $\mathbf{A}$ is a matrix with real eigenvalues only. All such systems have an equilibrium at the origin: we are interested in the behaviour of solutions near the origin, especially when viewed in phase space.

Example Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues of coefficient matrix are $\lambda = 3, -4$ with eigenvectors

$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively.

The general solution is:

Straight-line solutions are:

To see behaviour of solutions that are not straight-line solutions, i.e., solutions with $c_1 \neq 0$ and $c_2 \neq 0$, note that as $t \to \infty$

$$\mathbf{Y}(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

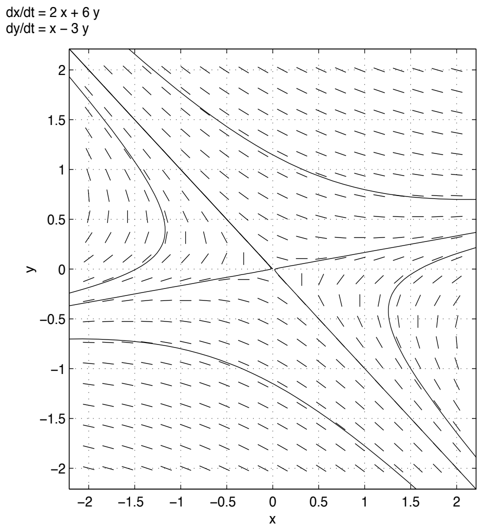
i.e., as $t \to \infty$, these solutions behave like the straight-line solution

$$c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

Similarly, as $t \to -\infty$, these solutions behave like the straight-line solution

$$c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Direction field and some solutions



Note that on solution curves for the straight-line solution

$$\mathbf{Y}_1(t) = c_1 e^{3t} \begin{pmatrix} 6 \\ 1 \end{pmatrix},$$

the arrows point away from the origin, indicating that $\mathbf{Y}_1(t) \to \mathbf{0}$ as $t \to -\infty$. Similarly, arrows on solution curves for the straight-line solution

$$\mathbf{Y}_2(t) = c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

point towards the origin, indicating that $\mathbf{Y}_2(t) \to \mathbf{0}$ as $t \to \infty$.

This example illustrates typical behaviour of solutions to a planar linear system with one positive real eigenvalue and one negative real eigenvalue. A characteristic feature of phase portrait is the presence of two special lines:

- On one line, solutions tend to origin as $t \to \infty$;
- On other line, solutions tend to origin as $t \to -\infty$.
- All other solutions tend to ∞ as $t \to \pm\infty$.

The equilibrium point at the origin in this type of system is called a **saddle**.

Example Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues of coefficient matrix are $\lambda = -5, -2$ with eigenvectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively.

The general solution is:

Straight-line solutions are:

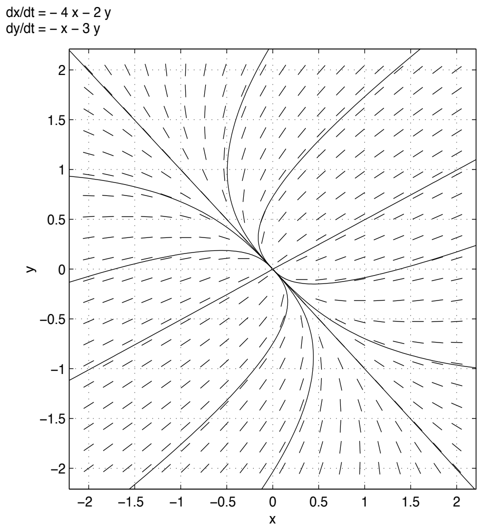
As $t \to \infty$, $e^{-5t} \to 0$ and $e^{-2t} \to 0$, so all solutions tend to the origin as $t \to \infty$.

This is a general result: if all eigenvalues of matrix $\mathbf{A}$ are real and negative, then all solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

tend to the origin as $t \to \infty$.

Direction field and some solutions:



This picture suggests that most solutions are tangent to the straight line solution

$$e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as $t \to \infty$. We can prove this:

Slope of solution curves is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

So, if $c_2 \neq 0$,

$$\lim_{t \to \infty} \left(\frac{dy}{dx} \right) = -1.$$

Thus as $t \to \infty$, all solutions tend to the origin and almost all are tangent to the straight-line solution

$$e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In general, in a system with 2 real negative eigenvalues, $\lambda_1 < \lambda_2 < 0$, all solutions tend to the origin as $t \to \infty$. Except for those solutions starting on the line of eigenvectors corresponding to λ_1, all solutions are tangent at $(0, 0)$ to the line of eigenvector corresponding to λ_2.

The equilibrium point in this type of system is called a **sink**.

Example Determine the behaviour of solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

Eigenvalues of coefficient matrix are $\lambda = 5, 2$ with eigenvectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively.

The general solution is:

Straight-line solutions are:

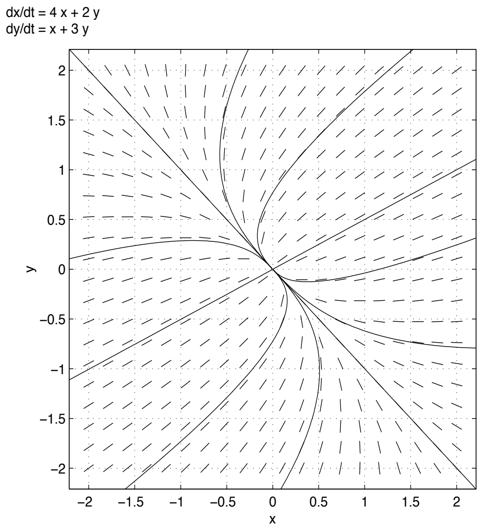
As $t \to \infty$, all non-zero solutions move away from the origin.

This is a general result: if all eigenvalues of $\mathbf{A}$ are real and positive, all non-zero solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

tend away from the origin as $t \to \infty$.

Direction field and some solutions:



This picture suggests that most solutions are tangent to the straight line solution

$$e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as $t \to -\infty$. We can prove this either

- by method used in last example, or
- by noting that this example corresponds to reversing time in the last example. Hence, phase portrait is the same as in last example but with direction of arrows reversed.

This is a general result. If $\mathbf{A}$ is a 2×2 matrix with eigenvalues λ_1 and λ_2, with $0 < \lambda_2 < \lambda_1$, then except for those solutions starting on the line of the eigenvectors corresponding to λ_1, all solutions are tangent at $(0,0)$ to the line of eigenvectors corresponding to λ_2.

The equilibrium point in this case is called a **source**.

This classification of equilibria extends to higher dimensions:

For the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY},$$

$\mathbf{Y} = \mathbf{0}$ is always an equilibrium. Assuming that all eigenvalues of $\mathbf{A}$ are real and distinct, then:

1. If all eigenvalues of $\mathbf{A}$ are positive, $\mathbf{Y} = \mathbf{0}$ is a **source**.
2. If all eigenvalues of $\mathbf{A}$ are negative, $\mathbf{Y} = \mathbf{0}$ is a **sink**.
3. If at least one eigenvalue of $\mathbf{A}$ is negative and at least one eigenvalue is positive, $\mathbf{Y} = \mathbf{0}$ is a **saddle**.