Maths 260 Lecture 18

Topic for today
Straight-line solutions

Reading for this lecture
BDH Section 3.2

Suggested exercises
BDH Section 3.2, 1, 5, 11, 13, 25

Reading for next lecture
BDH Section 3.3

Today's handout
Lecture 18 notes

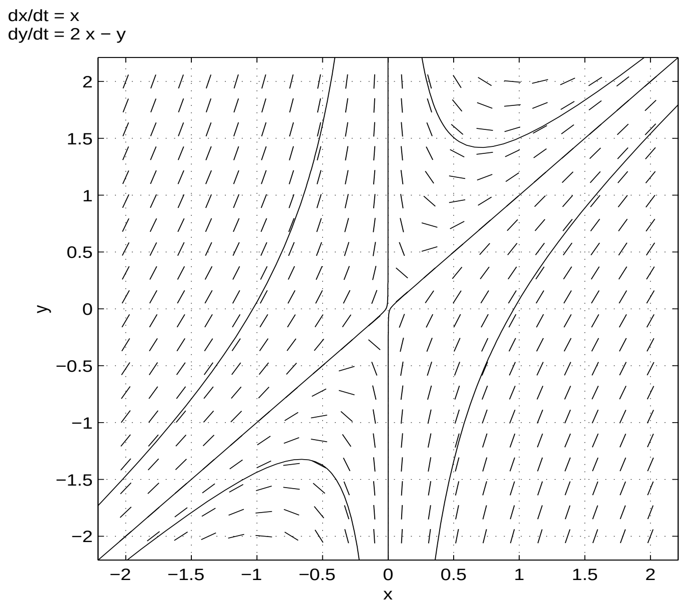
Section 2.6 Straight-Line Solutions

Given a system of DEs such as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix},$$

we want to find two linearly independent solution vectors that could be used to construct the general solution.

Direction field and some solutions:



Note straight-line solutions - these are linearly independent solutions. Can we find them?

To find a straight-line solution, note that at a point (x, y) on a straight-line solution, the vector field at that point must point in the same (or opposite) direction as the vector from the origin to (x, y).

This means

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{1}$$

where $\mathbf{v} = (x, y)$ and λ is a real number. If $\lambda > 0$, vector field points in same direction as $\mathbf{v}$, i.e., away from the origin. If $\lambda < 0$, vector field points in opposite direction to $\mathbf{v}$, i.e., towards the origin. Equation (1) says that $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ.

To get a formula for the straight-line solutions, write

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$$

where $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ. Then $\mathbf{Y}(t)$ is a solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$

Also, as t varies, $e^{\lambda t}$ just increases or decreases or remains constant (depending on λ) and $\mathbf{v}$ is constant so the solution curve for $\mathbf{Y}(t)$ is a straight line.

Summary

Can find a straight-line solution of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

by finding a real eigenvalue, λ, of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$; a straight-line solution is then

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$$

Example

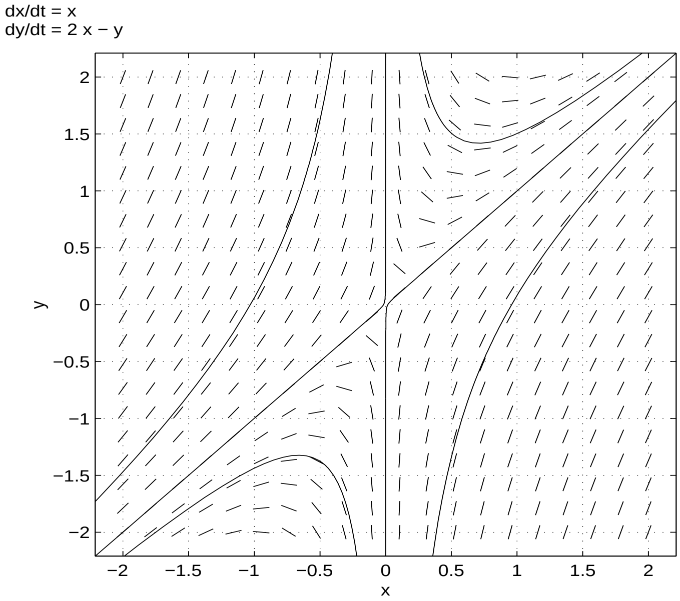
Find any straight-line solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

Direction field and some solutions



Note: In this example the two straight-line solutions are linearly independent. This is as expected because:

Result from linear algebra
Let λ_1 and λ_2 be two real and distinct eigenvalues for the matrix $\mathbf{A}$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Hence, the two straight-line solutions $\mathbf{Y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{Y}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$ are linearly independent at $t = 0$ and thus are linearly independent solutions.

Grand summary
If $\mathbf{A}$ is an $m \times m$ matrix with real eigenvalues $\lambda_1, ..., \lambda_k$ and corresponding eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_k$, then $\mathbf{Y}_1 = e^{\lambda_1 t}\mathbf{v}_1,\ ..., \mathbf{Y}_k = e^{\lambda_k t}\mathbf{v}_k$ are straight-line solutions of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$

Furthermore, if all the λ_i are distinct and $k = m$ (i.e., there are m real and distinct eigenvalues of A), then the set $\{\mathbf{Y}_1, ..., \mathbf{Y}_k\}$ is linearly independent and the general solution to the system is

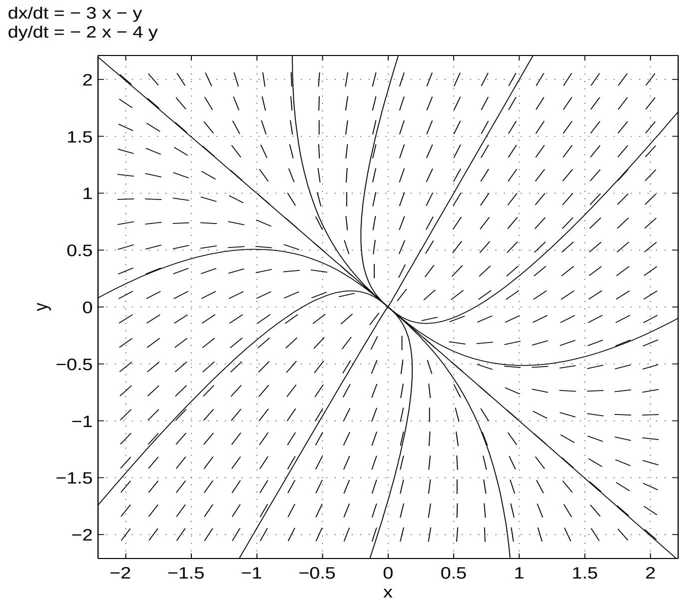
$$\mathbf{Y}(t) = c_1\mathbf{Y}_1 + ... + c_m\mathbf{Y}_m.$$

Example 1 : Find the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{A} = \begin{pmatrix} -3 & -1 \\ -2 & -4 \end{pmatrix}.$$

Describe the longterm behaviour of solutions.

Direction field and some solutions



Example 2 : Find the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \mathbf{A} = \begin{pmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Describe the longterm behaviour of solutions.

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